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ALMOST GEODESIC MAPPINGS OF THE FIRST TYPE ONTO SYMMETRIC SPACES

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Abstract. The article is devoted to the theory of almost geodesic mappings of the first type onto symmetric spaces. There are found certain necessary and sufficient conditions when a space with affine connection admits a canonical almost geodesic mapping of the first type onto symmetric space.

Keywords: Almost geodesic mapping, symmetric spaces

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1 Introduction

The article deals with further study of theory of almost geodesic mappings of spaces with affine connection. The idea of this theory is based on the article [13] of Levi-Civita. He has formulated and solved (in a special coordinate system) a task to find Riemannian spaces with general geodesics.

Let us remark, that it was related with study of equations of mechanical systems. Subsequently, the theory of geodesic mappings has been developed by Thomas, Weyl, Shirokov, Solodovnikov, Sinyukov, Mikeš and others. Some questions which has appeared during studying of geodesic mappings were developed by Kagan, Vrançeanu, Shapiro and others. Above authors have found special classes of $(n-2)$-projective spaces. See [15, 16, 22].

A.Z. Petrov [19] has introduced a notion of quasi geodesic mappings. In particular, special quasi geodesic mappings are holomorphically projective mappings of Kähler spaces. These mappings were first studied by Otsuki, Tashiro, Prvanovich, Sakaguchi, Mikeš, and others, see [15, 16, 14, 17, 18, 22].

A natural generalization of these classes of mappings is a notion of *almost geodesic mappings* which has been introduced by Sinyukov [22]. He introduced three types of almost geodesic mappings: π_1, π_2 and π_3. See also [2, 3, 18], [15] (pp. 455–462).

We note, that to a theory of almost geodesic mappings π_1 is devoted many papers, for example [1, 4, 5, 6, 7, 9, 8, 10, 18], see also in books [22] and [15] (pp. 463–480).

Fundamental equations of canonical almost geodesic mappings of the first type of spaces with affine connection onto symmetric spaces are described in [22] in the form of closed Cauchy system in covariant derivatives. In this book there is determined a set of basic parameters on which a general solution of such system depends.

Let us consider a simply-connected space the dimension n of which is greater than 2. Suppose that geometric objectives are continuous and sufficiently smooth.

2 Fundamental notions of the theory of almost geodesic mappings of spaces with affine connection

Let us remind basic notions and theorems of theory of almost geodesic mappings of spaces with affine connection as presented by Sinyukov in [22], see [15] (pp. 455–480).

We consider a space A_n with affine connection ∇ without torsion, which is related to a local coordinate system $x^1, x^2, \dots, x^n$ with components $\Gamma^h_{ij}(x)$ of connection ∇. A curve ℓ: $x^h = x^h(t)$ in a space with affine connection A_n, $n > 2$, is called an *almost geodesic* if its tangent vector $\lambda^h = dx^h(t)/dt$ fulfils equalities

$$\lambda^h_2 = a(t)\,\lambda^h + b(t)\,\lambda^h_1 \tag{1}$$

where $\lambda^h_1 \equiv \lambda^h_{,\alpha}\lambda^\alpha$, $\lambda^h_2 \equiv \lambda_{1,\alpha}\lambda^\alpha$, comma denotes covariant derivative respective connection ∇ of space A_n, $a(t)$ and $b(t)$ are some functions of a said argument.

The mapping π of a space A_n with affine connection ∇ onto a space $\bar{A}_n$ with affine connection $\bar{\nabla}$ is called an *almost geodesic mapping* if every geodesic in space A_n is mapped onto an almost geodesic in space $\bar{A}_n$.

Theorem 1. (Sinyukov [22]) *A mapping A_n onto $\bar{A}_n$ is almost geodesic if and only if in a common coordinate system $x^1, x^2, \dots, x^n$ with respect to this mapping a connection deformation tensor*

$$P^h_{ij}(x) = \bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x) \tag{2}$$

identically fulfils with respect to the $x^1, x^2, \dots, x^n$ and $\lambda^1, \lambda^2, \dots, \lambda^n$ the following condition

$$(P^h_{\alpha\beta,\gamma} + P^h_{\delta\alpha}P^\delta_{\beta\gamma})\lambda^\alpha\lambda^\beta\lambda^\gamma = b\,P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + a\,\lambda^h \tag{3}$$

where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are components of affine connection on A_n and $\bar{A}_n$, $\lambda^1, \lambda^2, \dots, \lambda^n$ are components of some vector, a and b are functions depending on $x^1, x^2, \dots, x^n$ and $\lambda^1, \lambda^2, \dots, \lambda^n$.

Relating to character of a dependency of functions a and b on $\lambda^1, \lambda^2, \dots, \lambda^n$ Sinyukov has defined three types of almost geodesic mappings π_1, π_2 and π_3.

We have proved (see [2, 3], [15] (p. 459)) that in the case $n > 5$ there does not exists any other type of geodesic mapping.

Especially, a mapping π_1: $A_n \to \bar{A}_n$ is a almost geodesic mapping of the type π_1 if in a common coordinate system with respect to π_1 the following conditions are fulfil

$$P^h_{(ij,k)} + P^\alpha_{(ij}P^h_{k)\alpha} = \delta^h_{(i}a_{jk)} + b_{(i}P^h_{jk)} \tag{4}$$

where a_{ij} is a symmetric tensor, b_i is some covector and brackets denote a symmetrization with respect to denoted indices.

If a covector b_i is identically equal to zero then the mappings is called *canonical almost geodesic mapping of the type* π_1.

It is known (see [22]) that any almost geodesic mapping of the type π_1 may be represented in the form of a composition of canonical geodesic mapping of the type π_1 and a geodesic mapping.

3 Diffeomorphisms of spaces with affine connection onto symmetric spaces

A space with affine connection $\bar{A}_n$ is called (*local*) *symmetric*, if the Riemann tensor is absolutely parallel (P.A. Shirokov [20, 21], É. Cartan [11], S. Helgason [12]). By this way, symmetric spaces $\bar{A}_n$ are characterized by

$$\bar{R}^h_{ijk|m}(x) \equiv 0 \tag{5}$$

where $\bar{R}^h_{ijk}$ is the Riemannian tensor of $\bar{A}_n$, symbol "|" denotes covariant derivative with respect to connection $\bar{\nabla}$ of a space $\bar{A}_n$.

By diffeomorphism of spaces with affine connection we mean a bijective smooth mapping, the inverse of which is also a smooth mapping. Almost geodesic mappings have an important role between diffeomorphisms of spaces with affine connection.

Let us suppose that a space A_n with affine connection ∇ admits a diffeomorphism f onto a space $\bar{A}_n$ with affine connection $\bar{\nabla}$ and these mappings are related to a common coordinate system $(x^1, x^2, \ldots, x^n)$ with respect to the mapping f. For covariant derivative of the Riemannian tensor $\bar{R}^h_{ijk}$ of space $\bar{A}_n$ with affine connection $\bar{\nabla}$ we have

$$\bar{R}^h_{ijk|m} = \frac{\partial \bar{R}^h_{ijk}}{\partial x^m} + \bar{\Gamma}^h_{m\alpha}\bar{R}^\alpha_{ijk} - \bar{\Gamma}^\alpha_{mi}\bar{R}^h_{\alpha jk} - \bar{\Gamma}^\alpha_{mj}\bar{R}^h_{i\alpha k} - \bar{\Gamma}^\alpha_{mk}\bar{R}^h_{ij\alpha}. \tag{6}$$

Considering (2) we may formula (6) written in the form:

$$\bar{R}^h_{ijk|m} = \bar{R}^h_{ijk,m} + P^h_{m\alpha}\bar{R}^\alpha_{ijk} - P^\alpha_{mi}\bar{R}^h_{\alpha jk} - P^\alpha_{mj}\bar{R}^h_{i\alpha k} - P^\alpha_{mk}\bar{R}^h_{ij\alpha}. \tag{7}$$

In what follows we suppose that a space $\bar{A}_n$ is symmetric. With respect to the formula (5) we obtain from (7):

$$\bar{R}^h_{ijk,m} = -P^h_{m\alpha}\bar{R}^\alpha_{ijk} + P^\alpha_{mi}\bar{R}^h_{\alpha jk} + P^\alpha_{mj}\bar{R}^h_{i\alpha k} + P^\alpha_{mk}\bar{R}^h_{ij\alpha}. \tag{8}$$

Formulas (8) hold for diffeomorphisms of any nature of spaces with affine connection onto symmetric spaces respective to common coordinate system $(x^1, x^2, \ldots, x^n)$.

4 Canonical almost geodesic mappings of the first type of spaces with affine connection onto symmetric spaces

Let us consider a canonical almost geodesic mappings of spaces A_n with affine connection onto symmetric spaces $\bar{A}_n$. It is well known (see [22], [15] (p. 463)) that equations (4) may be expressed in the form

$$3\,(P^h_{ij,k} + P^\alpha_{ij}P^h_{\alpha k}) = R^h_{(ij)k} - \bar{R}^h_{(ij)k} + \delta^h_{(k}a_{ij)} + b_{(i}P^h_{jk)} \tag{9}$$

where R^h_{ijk} is the Riemannian tensor of a space A_n.

Therefore canonical almost geodesic mappings of the first type of spaces with affine connection will be characterized by equations

$$P^h_{ij,k} = -P^\alpha_{ij} P^h_{\alpha k} + \frac{1}{3}\,(R^h_{(ij)k} - \bar{R}^h_{(ij)k} + \delta^h_{(k} a_{ij)}). \tag{10}$$

Covariant differentiate (10) with respect to x^m. Considering formulas (8) and (10) we may derive:

$$\begin{aligned} P^h_{ij,km} = &-\tfrac{1}{3}\,R^\alpha_{(ij)m} P^h_{\alpha k} - \tfrac{1}{3}\,R^h_{(\alpha k)m} P^\alpha_{ij} + \tfrac{1}{3}\,R^h_{(ij)k,m} + P^\beta_{ij} P^\alpha_{\beta m} P^h_{\alpha k} + \\ &P^\beta_{\alpha k} P^h_{\beta m} P^\alpha_{ij} + \tfrac{1}{3}\,(\bar{R}^\alpha_{(ij)m} P^h_{\alpha k} + \bar{R}^h_{(\alpha k)m} P^\alpha_{ij} + P^h_{m\alpha} \bar{R}^\alpha_{(ij)k} - P^\alpha_{mk} \bar{R}^h_{(ij)\alpha} - \\ &P^\alpha_{mi} \bar{R}^h_{(j\alpha)k} - P^\alpha_{mj} \bar{R}^h_{(i\alpha)k} - \delta^\alpha_{(m} a_{ij)} P^h_{\alpha k} - \delta^h_{(\alpha} a_{km)} P^\alpha_{ij} + \delta^h_{(k} a_{ij),m}). \end{aligned} \tag{11}$$

Alternating (11) with respect to k and m and using Ricci identity and properties Riemannian tensor we may obtain:

$$\begin{aligned} \delta^h_{(m} a_{ij),k} - \delta^h_{(k} a_{ij),m} = &-3P^h_{\alpha j} R^\alpha_{ikm} - 3P^h_{i\alpha} R^\alpha_{jkm} - R^\alpha_{(ij)m} P^h_{\alpha k} + R^\alpha_{(ij)k} P^h_{\alpha m} + \\ &R^h_{(ij)k,m} - R^h_{(ij)m,k} + 3\bar{R}^h_{\alpha km} P^\alpha_{ij} - P^\alpha_{mi} \bar{R}^h_{(j\alpha)k} + P^\alpha_{ki} \bar{R}^h_{(j\alpha)m} - P^\alpha_{mj} \bar{R}^h_{(i\alpha)k} + \\ &P^\alpha_{kj} \bar{R}^h_{(i\alpha)m} - \delta^\alpha_{(m} a_{ij)} P^h_{\alpha k} + \delta^\alpha_{(k} a_{ij)} P^h_{\alpha m}. \end{aligned} \tag{12}$$

Contracting equation (12) with respect to indices h and m we may derive that

$$\begin{aligned} &(n+1)\,a_{ij,k} - a_{ik,j} - a_{jk,i} = \\ &-3P^\beta_{\alpha(j} R^\alpha_{i)k\beta} - P^\beta_{\alpha k} R^\alpha_{(ij)\beta} + P^\beta_{\alpha\beta} R^\alpha_{(ij)k} + R^\beta_{(ij)k,\beta} - R_{(ij),k} + 3P^\alpha_{ij} \bar{R}_{\alpha k} - \\ &P^\alpha_{\beta i} \bar{R}^\beta_{(j\alpha)k} + P^\alpha_{ki} R_{(j\alpha)} - P^\alpha_{\beta j} \bar{R}^\beta_{(i\alpha)k} + P^\alpha_{kj} \bar{R}_{(i\alpha)} - \delta^\alpha_{(\beta} a_{ij)} P^\beta_{\alpha k} + a_{(ij} P^\beta_{k)\beta} \end{aligned} \tag{13}$$

where R_{ij}, $\bar{R}_{ij}$ are Ricci tensors od spaces A_n and $\bar{A}_n$, respectively.

Alternating equations (13) with respect indices j and k we may derive that

$$\begin{aligned} a_{ij,k} - a_{ik,j} = \tfrac{1}{n+2}\,\big[&P^\beta_{\alpha k}(R^\alpha_{ij\beta} + R^\alpha_{\beta ji}) - P^\beta_{\alpha j}(R^\alpha_{ik\beta} + R^\alpha_{\beta ki}) + 3P^\beta_{\alpha\beta} R^\alpha_{ijk} + \\ &3R^\beta_{ijk,\beta} - R_{(ij),k} + R_{(ik),j} + 2P^\alpha_{ij} \bar{R}_{\alpha k} - 2P^\alpha_{ik} \bar{R}_{\alpha j} + P^\alpha_{ki} \bar{R}_{j\alpha} - P^\alpha_{ji} \bar{R}_{k\alpha} - \\ &P^\alpha_{\beta j} \bar{R}^\beta_{(i\alpha)k} + P^\alpha_{\beta k} \bar{R}^\beta_{(i\alpha)j} - a_{ij} P^\alpha_{\alpha k} - a_{\alpha j} P^\alpha_{ik} + a_{ik} P^\alpha_{\alpha j} + a_{\alpha k} P^\alpha_{ij}\big]. \end{aligned} \tag{14}$$

Wit respect to equations (14) we may the equation (13) write in the form:

$$\begin{aligned} (n-1)\,a_{ij,k} = &-3P^\beta_{\alpha(j} R^\alpha_{i)k\beta} - P^\beta_{\alpha k} R^\alpha_{(ij)\beta} + P^\beta_{\alpha\beta} R^\alpha_{(ij)k} + R^\beta_{(ij)k,\beta} - R_{(ij)k} + \\ &3P^\alpha_{ij} \bar{R}_{\alpha k} - P^\beta_{\alpha i} \bar{R}^\alpha_{(j\beta)k} + P^\alpha_{ki} \bar{R}_{(j\alpha)} - P^\beta_{\alpha j} \bar{R}^\alpha_{(i\beta)k} + P^\alpha_{kj} \bar{R}_{(i\alpha)} - \\ &\delta^\alpha_{(\beta} a_{ij)} P^\beta_{\alpha k} + \delta^\alpha_{(k} a_{ij)} P^\beta_{\alpha\beta} - \tfrac{1}{n+2}\,B_{(ij)k} \end{aligned} \tag{15}$$

where

$$B_{ijk} = P^{\beta}_{\alpha k}(R^{\alpha}_{ij\beta} + R^{\alpha}_{\beta ji}) - P^{\beta}_{\alpha j}(R^{\alpha}_{ik\beta} + R^{\alpha}_{\beta ki}) + 3P^{\beta}_{\alpha\beta}R^{\alpha}_{ijk} + 3R^{\beta}_{ijk,\beta} -$$

$$R_{(ij),k} + R_{(ik),j} + 2P^{\alpha}_{ij}\bar{R}_{\alpha k} - 2P^{\alpha}_{ik}\bar{R}_{\alpha j} + P^{\alpha}_{ki}\bar{R}_{j\alpha} - P^{\alpha}_{ji}\bar{R}_{k\alpha} -$$

$$P^{\alpha}_{\beta j}\bar{R}^{\beta}_{(i\alpha)k} + P^{\alpha}_{\beta k}\bar{R}^{\beta}_{(i\alpha)j} - a_{ij}P^{\alpha}_{\alpha k} - a_{\alpha j}P^{\alpha}_{ik} + a_{ik}P^{\alpha}_{\alpha j} + a_{\alpha k}P^{\alpha}_{ij}.$$

Evidently, equations (8), (10) and (15) in a given space A_n present a form of closed Cauchy system with respect to functions $\bar{R}^h_{ijk}(x)$, $P^h_{ij}(x)$ and a_{ij}, which, naturally, must fulfil the following algebraic conditions

$$\bar{R}^h_{i(jk)} = \bar{R}^h_{(ijk)} = 0, \quad P^h_{ij} = P^h_{ji}, \quad a_{ij} = a_{ji}. \tag{16}$$

Theorem 2. *A space A_n with affine connection admits a canonical almost geodesic mapping of the type π_1 onto symmetric space $\bar{A}_n$ if and only if in A_n there exists a solution of mixed closed Cauchy system (8), (10), (15) and (16) with respect to functions $\bar{R}^h_{ijk}(x)$, $P^h_{ij}(x)$ and $a_{ij}(x)$.*

Evidently, general solution of a mixed closed Cauchy system (8), (10), (15) and (16) depends at most on $\frac{1}{6}\,n\,(n+1)\,(n^2+2n+3)$ real parameters.

Theorem 2 above precises results of theory of canonical almost geodesic mappings onto symmetric spaces which have been obtained in [22] and [8, 7].

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