

# On certain mappings onto Ricci symmetric manifolds

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## 1 On mappings onto Ricci symmetric manifolds

Let us recall that manifolds with affine connection and Riemannian manifolds are called *Ricci symmetric* if in them the Ricci tensor is absolutely parallel.

Let  $f: M_n \rightarrow \bar{M}_n$  be a diffeomorphism (possibly a bijection of “sufficiently high” differentiability class) between  $n$ -dimensional manifolds  $M_n$  and  $\bar{M}_n$ . The above considerations allow us to suppose that the manifolds in fact coincide,  $\bar{M} \equiv M$ .

Hence  $A_n = (M, \nabla)$  and  $\bar{A}_n = (\bar{M}, \bar{\nabla}) \equiv (\bar{M}, \nabla)$  be manifolds  $M$  and  $\bar{M}$  with affine connections  $\nabla$  and  $\bar{\nabla}$ , respectively. Then the type (1,2) tensor field

$$P = \bar{\nabla} - \nabla$$

is called the *deformation tensor* of the connections  $\nabla$  and  $\bar{\nabla}$  with respect to  $f$  on  $M$ .

We proved that  $A_n$  admits mapping  $f$  onto Ricci symmetric manifolds  $\bar{A}_n$  if and only if it satisfies the following equation

$$\nabla_m \bar{R}_{ij} = P_{mi}^\alpha \bar{R}_{\alpha j} + P_{mj}^\alpha \bar{R}_{i\alpha},$$

where  $\bar{R}_{ij}$  are components of the Ricci tensor on  $\bar{A}_n$ .

Here we considered certain special mappings onto Ricci symmetric manifolds.

## 2 Conformal mappings onto Ricci symmetric manifolds

The following Theorem 2.1 ([1]) is devoted to conformal mappings onto Ricci symmetric manifolds. These results are practically generalized results which we have obtained for conformal mappings of Riemannian manifolds onto Einstein spaces [3, 5] and geodesic mappings of manifolds with affine connection onto symmetric Riemannian manifolds [4, 6, 7].

**Theorem 2.1.**  *$n$ -dimensional (pseudo-) Riemannian manifold  $V_n$  admits conformal mapping onto Ricci symmetric (pseudo-) Riemannian manifold  $\bar{V}_n$  if and only if on  $V_n$  exists a solution of the following closed Cauchy type equations system in covariant derivative respective unknown functions  $\psi(x)$ ,  $\psi_i(x)$ ,  $\mu(x)$ , and  $\bar{R}_{ij}(x)$  ( $= \bar{R}_{ji}(x)$ ):*

$$\begin{aligned}\psi_{,i} &= \psi_i \\ \psi_{i,j} &= \frac{\mu}{n-2} g_{ij} + \psi_i \psi_j - \frac{1}{n-2} (\bar{R}_{ij} - R_{ij}), \\ \bar{R}_{i,j,k} &= 2\psi_k \bar{R}_{ij} + \psi_i \bar{R}_{jk} + \psi_j \bar{R}_{ik} - \psi^\alpha \bar{R}_{i\alpha} g_{jk} - \psi^\alpha \bar{R}_{j\alpha} g_{ik}, \\ (n-1)\mu_{,k} &= g^{\alpha\beta} ((n-2)\psi_\gamma R_{\beta k \alpha}^\gamma - (n-1)\psi_\beta \bar{R}_{\alpha k} - \psi_\beta R_{\alpha k}) + (R + (n-1)\mu) \psi_k - \frac{R_{,k}}{2},\end{aligned}$$

where comma denotes covariant derivative on  $V_n$ .

Here functions  $\bar{R}_{ij}(x)$  are components of the Ricci tensor on manifold  $\bar{V}_n$ .

## 3 Geodesic mappings onto Ricci symmetric manifolds

Analogical results were obtained for geodesic mappings [2].

**Theorem 3.1.**  *$n$ -dimensional manifold  $A_n$  with affine connection admits geodesic mapping onto Ricci symmetric manifold  $\bar{A}_n$  with affine connection if and only if on  $A_n$  exists a solution of the following closed Cauchy type equations system in covariant derivative respective unknown functions  $\psi_i(x)$  and  $\bar{R}_{ij}(x)$*

$$\begin{aligned}\bar{R}_{i,j,m} &= 2\psi_m \bar{R}_{ij} + \psi_i \bar{R}_{mj} + \psi_j \bar{R}_{im}, \\ \psi_{i,j} &= \frac{1}{n^2-1} (n\bar{R}_{ij} + \bar{R}_{ji} - (nR_{ij} + R_{ji})) + \psi_i \psi_j,\end{aligned}$$

where comma denotes covariant derivative on  $A_n$ .

Here functions  $\bar{R}_{ij}(x)$  are components of the Ricci tensor on manifold  $\bar{A}_n$ .

We set a number of principian parameters on which depend the general solutions of the Cauchy type equation systems in Theorems 2.1 and 3.1.

It is interesting to note that if Ricci symmetric manifold  $\bar{A}_n$  is equiaffine (i.e., the Ricci tensor is symmetric), then integrability condition of equations in Theorem 3.1 is linear respective unknown functions  $\psi_i(x)$  and  $\bar{R}_{ij}(x)$ .

## References

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