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COMPLEX SUBMANIFOLDS OF LCK-MANIFOLDS, PSEUDO-VAISMAN AND VAISMAN MANIFOLDS

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Abstract. The article is dedicated to the immersions of submanifolds in LCK-manifolds that a tangent space in all points of the submanifolds to be normal to Lee field and we find conditions under which LCK-manifold admits the immersion of complex submanifolds. Also we explore properties of Lee form of Vaisman and pseudo-Vaisman manifold.

Keywords: Complex manifold, submanifold, LCK-manifold, almost complex structure, Lee form

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1 Introduction

Differential geometric aspects of submanifolds of manifolds with certain structures are very fruitful fields for Riemannian geometry. Study of complex submanifolds immersed in locally conformal Kähler manifolds (for brevity, LCK-manifolds) was begun by Vaisman in [14], and more attention was paid to so called Generalized Hopf manifolds. Special mappings of Kähler and conformal Kähler manifolds were studied in [7, 8, 10, 11]. Certain questions on conformal mappings and conformal tensor are solved in [5, 6, 9, 12, 13].

We continue to study the immersions of submanifolds that a tangent space in all points of the submanifolds to be normal to Lee field. Also we explore properties of Lee form of Vaisman and pseudo-Vaisman manifolds.

2 Preliminaries

A Hermitian manifold (M^{2m}, J, g) is called a *locally conformal Kähler manifold* (*LCK-manifold*) if there is an open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ of M^{2m} and a family $\{\sigma_\alpha\}_{\alpha\in A}$ of C^∞ functions $\sigma_\alpha : U_\alpha \to \mathbb{R}$ so that each local metric

$$\hat{g}_\alpha = e^{-2\sigma_\alpha} g|_{U_\alpha}$$

is Kählerian. An LCK-manifold is endowed with some form ω, so called a *Lee form* which can be calculated as [2]

$$\omega = \frac{1}{m-1}\delta\Omega \circ J \quad \text{or} \quad \omega_i = -\frac{2}{n-2}J^{\alpha}_{\beta,\alpha}J^{\beta}_{i}, \tag{1}$$

The form should be closed:

$$d\omega = 0.$$

One can compute covariant derivative an almost complex structure with respect of the Levi-Civita connection of (M^{2m}, J, g) using the formulae

$$J^k_{i,j} = \frac{1}{2}\big(\delta^k_j J^{\alpha}_i \omega_{\alpha} - \omega^k J_{ij} - J^k_j \omega_i + J^k_{\alpha}\omega^{\alpha} g_{ij}\big). \tag{2}$$

Let (M^{2m}, J, g) be a complex m-dimensional Hermitian manifold, g is its Hermitian metric, J is its comlex structure. Consider an immersion of an m-dimensional manifold $\bar{M}^k$ in M^{2m}:

$$\Psi : \bar{M}^k \longrightarrow M^{2m}.$$

Let ∇ and $\overline{\nabla}$ be operators of covariant differentiations on M^{2m} and $\overline{M}^k$, respectively. Then the Gauss and Weingarten formulas are given by [1, p. 2] :

$$\nabla_X Y = \bar{\nabla}_X Y + h(X,Y), \tag{3}$$

$$\nabla_X \xi = -A_{\xi}X + \nabla^{\perp}_X \xi, \tag{4}$$

respectively, where X and Y are vector fields tangent to $\overline{M}^k$ and ξ normal to $\overline{M}^k$. $h(X,Y)$ is the *second fundamental form*, $\nabla^{\perp}$ the linear connection induced in the normal bundle $E(\Psi)$, called the *normal connection*, and A_{ξ} the *second fundamental tensor* at ξ.

We call $\overline{M}^k$ $CR-$ *submanifold* of (M^{2m}, J, g) if $\overline{M}^k$ carries a C^{∞} distribution D so that

1. D is *holomorphic* (i.e. $J_x(D_x) = D_x$) for any $x \in \overline{M}^k$,
2. the orthogonal complement $D^{\perp}$ with respect to $\overline{g} = \Psi^* g$ of D in $T(\overline{M}^k)$ is *anti-invariant* (i.e. $J_x({D^{\perp}}_x) \subseteq E(\Psi)_x$) for any $x \in \overline{M}^k$)[2, p. 153]

Let $(\overline{M}^k, D)$ be a $CR-$submanifold of the Hermitian manifold M_0^{2m}. Set $p = \dim_{\mathbb{C}} D_x$ and $p = \dim_{\mathbb{R}} D^{\perp}_x$; for any $x \in \overline{M}^k$ such that $2p + q = k$. If $q = 0$ then $\overline{M}^k$ is a *complex submanifold*, i.e. it is a complex manifold and Ψ is a holomorphic immersion. If $p = 0$ then $\overline{M}^k$ is an *anti-invariant submanifold* (i.e. $J_x(T_x(\overline{M}^k)) \subseteq E(\Psi)_x$ for any $x \in \overline{M}^k$). A $CR-$submanifold $(\overline{M}^k, D)$ is *proper* if $p \neq 0$ and $q \neq 0$. Also $(\overline{M}^k, D)$ is *generic* if $q = 2m - k$ (i.e. $J_x(T_x(\overline{M}^k)) = E(\Psi)_x$ for any $x \in \overline{M}^k$). A submanifold $\overline{M}^k$ of the complex manifold (M^{2m}, J) is *totally real* if

$$T_x(\overline{M}^k) \cap J_x(T_x(\overline{M}^k)) = \{0\}$$

for any $x \in \overline{M}^k$

3 Complex hypersurfaces of LCK-manifolds

Let submanifold $\overline{M}^k$ is immersed in LCK-manifold M^{2m}

$$\Psi : \bar{M}^k \longrightarrow M^{2m},$$

so that $k = 2p$ and for any $x \in \overline{M}^{2p}$

We are concerned with finding conditions under which LCK-manifold M^{2m} admits immersion of complex submanifolds. Then we obtain the following Theorem.

Theorem 1 *LCK-manifold M^{2m} admits immersion of complex hypersurface $\overline{M}^{2m-2}$ such that the Lee field $B = \omega^{\#}$ and the anti-Lee field $A = -JB = -J\omega^{\#}$ are normal to the hypersurface $\overline{M}^{2m-2}$ if and only if the Lee form of M^{2m} satisfies the condition*

$$\Phi_4(\nabla_X \omega(Y)) = \frac{\|\omega\|^2}{2} g(X, Y).$$

Here Φ_4 is the fourth Obata's projector:

$$\Phi_4(\omega_{i,j}) = \frac{1}{2}(\delta_i^a \delta_j^b + J_i^a J_j^b)\omega_{a,b}.$$

Necessity. Let us consider an LCK-manifold M^{2m}. Let $\theta = \omega \circ J$ and $A = -JB$ be respectively the anti-Lee form and the anti-Lee vector field. Then, we can rewrite (2) as

$$\nabla_X(J)Y = \frac{1}{2}\big(\theta(Y)X - \omega(Y)JX - g(X, Y)A - \Omega(X, Y)B\big),$$

and hence we get

$$\nabla_X A = -J\nabla B + \frac{1}{2}\big(\|\omega\|^2 JX + \omega(X) - \theta(X)B\big)$$

for any $X \in T(M^{2m})$. Let M^{2m-2} be a complex hypersurface of an M^{2m}. If $B \in E(\Psi)$, then $A \in E(\Psi)$ since the immersion is analytic one. Moreover, if $X, Y \in T(M^{2m-2})$, then $[X, Y] \in T(M^{2m-2})$ according to the classical Frobenius theorem. Hence

$$\begin{aligned} 0 = g([X, Y], A) &= g(\nabla_X Y, A) - g(\nabla_Y X, A) \\ &= -g(Y, \nabla_X A) + g(X, \nabla_Y A) = \\ = g(Y, J\nabla_X B) &- g(X, J\nabla_Y B) + \|\omega\|^2 \Omega(X, Y) \end{aligned} \tag{5}$$

Rewriting (5) in local coordinates, we obtain

$$\omega_{t,j} J_i^t - \omega_{t,i} J_j^t - \|\omega\|^2 J_{ij} = 0. \tag{6}$$

Next, multiply (6) by J_k^j:

$$\omega_{t,j} J_i^t J_k^j + \omega_{k,i} - \|\omega\|^2 g_{ik} = 0. \tag{7}$$

We can rewrite (7) as

$$2\Phi_4(\omega_{i,j}) - \|\omega\|^2 g_{ij} = 0.$$

where Φ_4 is the fourth Obata's projector [4]. For instance, applying the operator to a tensor Q^h_{ij} means

$$\Phi_4(Q^h_{ij}) = \frac{1}{2}(\delta^a_i\delta^b_j + J^a_i J^b_j)Q^h_{ab}.$$

Hence

$$\Phi_4(\omega_{i,j}) = \frac{\|\omega\|^2}{2}g_{ij}. \tag{8}$$

Sufficiency. Tangent bundle $T(M^{2m})$ should satisfy the system since the bundle is normal to both Lee field B and anti-Lee field A

$$\begin{cases} \omega = 0 \\ \theta = 0. \end{cases} \tag{9}$$

According to the Frobenius theorem the system (9) is completely integrable if and only if both Lee-form and anti-Lee form identically satisfy the conditions

$$\begin{array}{ll} 1) & d\omega \wedge \omega \wedge \theta = 0 \\ 2) & d\theta \wedge \omega \wedge \theta = 0. \end{array} \tag{10}$$

Identity (10_1) is satisfied since an M^{2m} is LCK-manifold, hence $d\omega = 0$. We have to explore (10_2). Let us take the exterior differential of the anti-Lee form $\theta = \omega \circ J$ [15, p. 6].

$$\begin{aligned} d\theta &= \frac{1}{2}\big(\nabla_k(\omega_i J^i_j) - \nabla_j(\omega_i J^i_k)\big)dx^k \wedge dx^j \\ &= \frac{1}{2}\big(\omega_{i,k}J^i_j + \omega_i J^i_{j,k} - \omega_{i,j}J^i_k - \omega_i J^i_{k,j}\big)dx^k \wedge dx^j \end{aligned}$$

According to (2) we obtain:

$$\begin{aligned} d\theta &= \big(\omega_{i,k}J^i_j + \omega_i J^i_{j,k}\big)dx^k \wedge dx^j \\ &= \Big(\omega_{i,k}J^i_j + \omega_i\frac{1}{2}\big(\delta^i_k J^t_j\omega_t - \omega^i J_{jk} - J^i_k\omega_j + J^i_t\omega^t g_{jk}\big)\Big)dx^k \wedge dx^j \\ &= \Big(\omega_{i,k}J^i_j + \frac{1}{2}\omega_k J^t_j\omega_t - \frac{1}{2}\|\omega\|^2 J_{jk} - \frac{1}{2}\omega_t J^t_k\omega_j\Big)dx^k \wedge dx^j \\ &= \frac{1}{2}\big(\omega_{i,k}J^i_j - \omega_{i,j}J^i_k - \|\omega\|^2 J_{jk} + \omega_k J^t_j\omega_t - \omega_t J^t_k\omega_j\big)dx^k \wedge dx^j \end{aligned}$$

Then,

$$\begin{aligned} d\theta \wedge \omega \wedge \theta &= \frac{1}{2}\big(\omega_{i,k}J^i_j - \omega_{i,j}J^i_k - \|\omega\|^2 J_{jk} \\ &\quad + \omega_k J^t_j\omega_t - \omega_t J^t_k\omega_j\big)dx^k \wedge dx^j \wedge \omega_l dx^l \wedge \omega_s J^s_h dx^h \\ &= \frac{1}{2}\big(\omega_{i,k}J^i_j - \omega_{i,j}J^i_k - \|\omega\|^2 J_{jk}\big)dx^k \wedge dx^j \wedge \omega_l dx^l \wedge \omega_s J^s_h dx^h, \end{aligned} \tag{11}$$

since the equation

$$\frac{1}{2}\big(\omega_k J^t_j\omega_t - \omega_t J^t_k\omega_j\big)dx^k \wedge dx^j \wedge \omega_l dx^l \wedge \omega_s J^s_h dx^h = 0;$$

is identically satisfied. Hence the equation

$$\omega_{i,k}J^i_j - \omega_{i,j}J^i_k - \|\omega\|^2 J_{jk} = 0,$$

gives us a sufficient condition for the identity $d\theta \wedge \omega \wedge \theta = 0$ to be satisfied. The condition coincides with (6) which is equivalent to (8):

$$\Phi_4(\omega_{i,j}) = \frac{1}{2}\|\omega\|^2 g_{ij}.$$

Hence (10_2) is satisfied too. Sufficiency is proved.

Taking into account that LCK-manifolds with Lee form which satisfies the condition

$$\Phi_4(\nabla\omega(X,Y)) = \frac{\|\omega\|^2}{2} g(X,Y) \tag{12}$$

have very particular properties, we propose call such LCK-manifolds as the *Pseudo-Vaisman manifolds*.

4 Pseudo-Vaisman manifolds

Let us discover the condition (12). An LCK-manifold is a Hermitian since its almost complex structure is integrable. This means the existence of the coordinate system in which J and metric g take correspondingly the forms

$$(J^i_j) = \begin{pmatrix} \sqrt{-1}\delta^\kappa_\lambda & 0 \\ 0 & -\sqrt{-1}\delta^{\hat\kappa}_{\hat\lambda} \end{pmatrix}, \qquad (g_{ij}) = \begin{pmatrix} 0 & g_{\mu\hat\lambda} \\ g_{\hat\mu\lambda} & 0 \end{pmatrix}.$$

In the system of coordinates we can rewrite (12) as

$$\omega_{\alpha,\hat\beta} = \frac{\|\omega\|^2}{2} g_{\alpha\hat\beta}. \tag{13}$$

One knows that the components of the connection with respect to the Hermitian metric g_{ij} by formulas [15, p. 64]

$$\begin{aligned} &1)\ \Gamma^\kappa_{\mu\lambda} = \frac{1}{2}\, g^{\kappa\hat\rho}\big(\partial_\mu g_{\lambda\hat\rho} + \partial_\lambda g_{\mu\hat\rho}\big), \qquad \text{conj.}\\ &2)\ \Gamma^\kappa_{\mu\hat\lambda} = \frac{1}{2}\, g^{\kappa\hat\rho}\big(\partial_{\hat\lambda} g_{\mu\hat\rho} - \partial_{\hat\rho} g_{\mu\hat\lambda}\big), \qquad \text{conj.}\\ &3)\ \Gamma^\kappa_{\hat\mu\hat\lambda} = 0, \qquad \text{conj.} \end{aligned} \tag{14}$$

where conj. means that there exists a formula which is the complex conjugate of the formula written at the left.

For LCK-manifolds (14_2) takes the form

$$\Gamma^\kappa_{\mu\hat\lambda} = \frac{1}{2} g^{\kappa\hat\rho}\big(\omega_{\hat\lambda} g_{\mu\hat\rho} - \omega_{\hat\rho} g_{\mu\hat\lambda}\big) = \frac{1}{2}\big(\omega_{\hat\lambda}\delta^\kappa_\mu - \omega^\kappa g_{\mu\hat\lambda}\big), \qquad \text{(conj.)}. \tag{15}$$

Hence taking into account (15) we can rewrite (13) as

$$\omega_{\alpha,\hat\beta} = \partial_{\hat\beta}\omega_\alpha - \omega_\kappa\Gamma^\kappa_{\alpha\hat\beta} - \omega_{\hat\kappa}\Gamma^{\hat\kappa}_{\alpha\hat\beta} = \partial_{\hat\beta}\omega_\alpha - \omega_{\hat\beta}\omega_\alpha + \frac{\|\omega\|^2}{2} g_{\alpha\hat\beta} = \frac{\|\omega\|^2}{2} g_{\alpha\hat\beta}.$$

Then we obtain

$$\partial_{\hat\beta}\omega_\alpha - \omega_{\hat\beta}\omega_\alpha = 0. \tag{16}$$

Let us multiply the both sides (16) by $e^{-\sigma}$ where σ is a function such that $d\sigma = \omega$. Then

$$-e^{-\sigma}\partial_{\hat{\beta}}\omega_\alpha + e^{-\sigma}\omega_{\hat{\beta}}\omega_\alpha = \partial_{\hat{\beta}}\big(-e^{-\sigma}\omega_\alpha\big) = 0.$$

Finally, we obtain the equation

$$\frac{\partial^2\psi}{\partial z^{\hat{\beta}}\partial z^\alpha} = 0. \tag{17}$$

The general solution of (17) is given by

$$\psi = f(z) + f(\hat{z}),$$

where $f(z)$ is a some analytic function of coordinates $z^1, z^2, ...z^m$, and $f(\hat{z})$ is a complex conjugate of the function $f(z)$, since ψ must be a real number which is subject to the condition $Ref(z) > 0$. Hence the function σ determining the conformal mapping is

$$\sigma = \ln \ \frac{1}{f(z) + f(\hat{z})} \ ,$$

and Lee form is

$$\omega_\alpha = -\frac{f'_\alpha(z)}{f(z) + f(\hat{z})}, \quad \omega_{\hat{\alpha}} = -\frac{f'_{\hat{\alpha}}(\hat{z})}{f(z) + f(\hat{z})} \ .$$

Hence we obtain the theorem.

Theorem 2 *If the Lee form of an LCK M^{2m} satisfies the condition*

$$\Phi_4(\nabla_X\omega(Y)) = \frac{\|\omega\|^2}{2}g(X,Y).$$

then the form is

$$\omega_\alpha = -\frac{f'_\alpha(z)}{f(z) + f(\hat{z})}, \quad \omega_{\hat{\alpha}} = -\frac{f'_{\hat{\alpha}}(\hat{z})}{f(z) + f(\hat{z})},$$

where $f(z)$ is a some analytic function such that $Ref(z) > 0$.

One might conclude that it is easy to construct pseudo-Vaisman manifold because it seems sufficient the latter admits a Kählerian metric. But that is wrong because not always it is possible to find an analytic function such that $Ref(z) > 0$ in every manifold's point. In particular, there exists a theorem.

Theorem 3 *There is no compact pseudo-Vaisman manifold.*

Proof. Let M is an LCK-manifold such that its Lee form satisfies the condition

$$\Phi_4(\omega_{i,j}) = \frac{1}{2}\|\omega\|^2 g_{ij}.$$

Transvecting this with g^{ij}, we find

$$\nabla_i\omega^i = \frac{n}{2}\|\omega\|^2.$$

On the other hand, according to the Theorem of Green [15, p. 21]

$$\int\limits_{M^n} \nabla_i\omega^i d\tau = 0,$$

where $d\tau$ is the volume element

$$d\tau = \sqrt{g} d\xi^1 \wedge d\xi^2 \wedge ... \wedge d\xi^n.$$

In this case we obtain

$$\frac{n}{2} \int_{M^n} \|\omega\|^2 d\tau = 0,$$

that is impossible for an LCK-manifold. The theorem is proved.

5 Vaisman manifods and Kähler-Vaisman potential

Let us concider locally conformal Kähler manifolds with a parallel Lee form.

$$\nabla_j \omega_i = 0.$$

In particular, using so called a holomorphic coordinate system we have

$$\nabla_{\hat{\beta}} \omega_\alpha = 0. \tag{18}$$

Taking into account (15), we can rewrite (18) as

$$\omega_{\alpha,\hat{\beta}} = \partial_{\hat{\beta}} \omega_\alpha - \omega_\kappa \Gamma^\kappa_{\alpha\hat{\beta}} - \omega_{\hat{\kappa}} \Gamma^{\hat{\kappa}}_{\alpha\hat{\beta}} = \partial_{\hat{\beta}} \omega_\alpha - \omega_{\hat{\beta}} \omega_\alpha + \frac{\|\omega\|^2}{2} g_{\alpha\hat{\beta}} = 0.$$

We get

$$\partial_{\hat{\beta}} \omega_\alpha - \omega_{\hat{\beta}} \omega_\alpha = -\frac{\|\omega\|^2}{2} g_{\alpha\hat{\beta}}. \tag{19}$$

Multiplying the both sides (19) by $e^{-\sigma}$ where σ is a function such that $d\sigma = \omega$ we obtain

$$-e^{-\sigma} \partial_{\hat{\beta}} \omega_\alpha + e^{-\sigma} \omega_{\hat{\beta}} \omega_\alpha = \partial_{\hat{\beta}} \big(-e^{-\sigma} \omega_\alpha \big) = \frac{\|\omega\|^2}{2} e^{-\sigma} g_{\alpha\hat{\beta}}.$$

Finally we get

$$\partial_{\hat{\beta}} \partial_\alpha \big(e^{-\sigma} \big) = \frac{\|\omega\|^2}{2} \bar{g}_{\alpha\hat{\beta}},$$

or, since on a Vaisman manifold condition $\|\omega\|^2 = const$ holds,

$$\partial_{\hat{\beta}} \partial_\alpha \big(\frac{2}{\|\omega\|^2} e^{-\sigma} \big) = \bar{g}_{\alpha\hat{\beta}}, \tag{20}$$

where $\bar{g}_{\alpha\hat{\beta}} = e^{-\sigma} g_{\alpha\hat{\beta}}$ is a local Kählerian metric. The theorem follows from (20).

Theorem 4 *For any Vaisman manifold M^n there locally exists a function*

$$V(z, \hat{z}) = \frac{2}{\|\omega\|^2} e^{-\sigma},$$

which determines a local Kählerian metric.

The function $V(z, \hat{z}) = \frac{2}{\|\omega\|^2} e^{-\sigma}$ is said to be a *Kähler-Vaisman potential.*

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