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Geodesic Mappings onto Generalized *m*-Ricci-Symmetric Spaces

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Abstract: In this paper, we study geodesic mappings of spaces with affine connections onto generalized 2-, 3-, and m-Ricci-symmetric spaces. In either case, the main equations for the mappings are obtained as a closed system of linear differential equations of the Cauchy type in the covariant derivatives. For the systems, we have found the maximum number of essential parameters on which the solutions depend. These results generalize the properties of geodesic mappings onto symmetric, recurrent, and also 2-, 3-, and m-(Ricci-)symmetric spaces with affine connections.

Keywords: geodesic mapping; space with affine connections; m-Ricci-symmetric space; Cauchy-type differential equations

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1. Introduction

In 1865, Beltrami considered a special case of geodesic mappings, namely, geodesic mappings of spaces with constant curvature, and in 1869, Dini posed a more general problem of the existence of possible geodesic mappings between surfaces. Later, Levi–Civita [1] studied geodesic mappings between Riemannian spaces and derived the basic equations for the mappings. We note the remarkable fact that this problem is related to the study of equations for the dynamics of mechanical systems. Finally, Weyl [2] obtained these *Levi–Civita equations* for geodesic mappings between spaces with an affine connection.

The theory of geodesic mappings was developed by Weyl [2], T. Thomas [3], J. Thomas [4], Eisenhart [5], Shirokov [6], Solodovnikov [7], Petrov [8], Sinyukov [9], Aminova [10], Mikeš et al. [11–13], Stanković, Velimirović et al. [14,15], and others.

Sinyukov [16] started to study geodesic mappings of symmetric and recurrent spaces onto (pseudo-)Riemannian spaces. He proved that for non-trivial geodesic mappings, these spaces are only projective Euclidean. An analogical result was obtained by Fomin [17] for the geodesic mapping of infinite dimensional symmetric Riemannian spaces, and Hinterleitner and Mikeš [18] for the geodesic mapping of generalized recurrent spaces onto Weyl spaces.

Later, these results were generalized by Mikeš [19–21] and Sobchuk [22,23] for m-symmetric, m-recurrent, and also 1-, 2-, and m-Ricci-symmetric spaces. The above-mentioned results are in local form. The global results for these spaces were obtained in many papers—for example, by Sinyukova [24], Stepanov [25], and Mikeš [11–13,26–28].

We note that the Ricci tensor plays an important role in Einstein's fundamental equations of the general theory of relativity (Petrov [8]). In the m-symmetric and m-recurrent spaces, the gravitational waves (Kaigorodov [29,30]) arise. These spaces are characterized by high-order differential equations with components of the Ricci tensor. Their solution is often periodical and it creates the conditions for the formation of gravitational waves. The questions about symmetry and projective structures were studied by Hall [31].

For the geodesic mappings of spaces with affine connections onto generalized Ricci-symmetric spaces, there exist non-trivial results (i.e., the spaces are not projective Euclidean). In this case, the solutions are obtained as a closed system of partial differential equations (PDEs) of the Cauchy type in the covariant derivatives; see Berezovski et al. [32–34].

Many other geometric results have been formulated in the form of Cauchy-type PDEs, such as isometric, conformal, projective and holomorphically projective motions, mappings and deformations, see [8–13], and also, for example, for almost-geodesic and rotary mappings [35,36].

Finally, we note that even for geodesic mappings between Riemannian spaces, equations of the Cauchy type cannot always be found due to the insufficient differentiability of metrics. In this case, geodesic mappings preserve geodesics bifurcations, for example [37–39]. On the other hand, it was found that the differentiability of metrics is maintained in geodesic mappings [40,41].

The paper is devoted to the study of geodesic mappings of spaces with affine connections onto generalized 2-, 3-, and m-Ricci-symmetric spaces with affine connections. The main equations for the mappings were obtained as closed systems of partial differential equations of the Cauchy type in a covariant derivative.

We assume that all geometric objects under consideration are not only continuous but also sufficiently smooth.

2. Generalized m-Ricci-Symmetric Spaces

Let A_n be an n-dimensional space with an affine connection ∇. *Ricci-symmetric* spaces A_n are defined by the equation:

$$\nabla_k R_{ij} = 0$$

and they are a generalization of locally symmetric Riemannian spaces in the sense of É. Cartan, for which the equation $\nabla_l R^h_{ijk} = 0$ holds, where R^h_{ijk} and R_{ij} are components of the Riemann and Ricci tensors, respectively. In 1925, P.A. Shirokov proved that any Ricci-symmetric Riemannian space is locally a direct product of Einstein manifolds; see [6]. This structural theorem was extended to submanifolds of Euclidean spaces; see [42].

A generalization of Ricci-symmetric spaces A_n are manifolds whose Ricci tensor satisfies the Codazzi equation

$$\nabla_k R_{ij} = \nabla_j R_{ik}.$$

It is known (see [43]) that these equations are valid for the symmetric Ricci tensor of the projective Euclidean space A_n. Riemannian manifolds with such a Ricci tensor were studied by A. Gray and other geometers under the name of $\mathcal{B}$-spaces (see [44], pp. 436–440).

The papers [29,30] defined *2-symmetric spaces* of general relativity as four-dimensional pseudo-Riemannian spaces with the metric signature $(- + ++)$, for which the equation $\nabla_l \nabla_m R^h_{ijk} = 0$ holds. This allows us to conclude that 2-symmetric spaces are 2-Ricci symmetric spaces, which are characterized by the equation

$$\nabla_l \nabla_m R_{ij} = 0.$$

In the papers [29,30], there was found a connection between the equation $\nabla_l \nabla_m R^h_{ijk} = 0$ and the existence of gravitational plane waves. It was also stated in [29,30] that there are 2-symmetric spaces V_4 different from the locally symmetric spaces.

In [29,30], a generalization of 2-symmetric spaces in the form of *semi-symmetric spaces* was proposed, i.e., such (pseudo-)Riemannian spaces V_n, for which the following equality holds:

$$\nabla_l \nabla_m R^h_{ijk} = \nabla_m \nabla_l R^h_{ijk}.$$

These equations can also be written as $R(X,Y) \cdot R = 0$ for the curvature tensor R and any vector fields X and Y on A_n. Such spaces were studied by N.S. Sinyukov [9] and were called *semi-symmetric*.

Obviously, the above equations imply the equation:

$$\nabla_l \nabla_m R_{ij} = \nabla_m \nabla_l R_{ij}.$$

These equations can be written in the form $R(X,Y) \cdot Ric = 0$ for the Ricci tensor Ric and any vector fields X and Y on V_n. This allows us to conclude that semi-symmetric spaces are *Ricci semi-symmetric spaces*. This form appears in many papers on Riemannian differential geometry (see [45–50]).

The papers [29,30] also gave the definition of *m-symmetric* (pseudo-)Riemannian manifolds V_n, for which the equation

$$\nabla_{l_1} \nabla_{l_2} \cdots \nabla_m R^h_{ijk} = 0$$

is satisfied. Analogically, these spaces are *m-Ricci-symmetric* ones which satisfy the condition

$$\nabla_{l_1} \nabla_{l_2} \cdots \nabla_{l_m} R_{ij} = 0.$$

A formal generalization of this class of spaces is *generalized m-Ricci-symmetric spaces*, which are characterized by the condition

$$\nabla_{l_1} \nabla_{l_2} \cdots \nabla_{l_{m-1}} (\nabla_k R_{ij} + \nabla_j R_{ik}) = 0.$$

If, in a generalized m-Ricci-symmetric space A_n, the Ricci tensor is symmetric (i.e., A_n is equiaffine, including (pseudo-)Riemannian; see [9–13,43]), then this space is m-Ricci-symmetric. Moreover, the more general statement is true: *if, in a generalized m-Ricci-symmetric space, the tensor $\nabla_{l_1} \nabla_{l_2} \cdots \nabla_{l_m} R_{ij}$ is symmetric respective to the indices i and j, then this space is also m-Ricci-symmetric.* To prove this statement, we have to calculate the following:

$$\begin{aligned} \nabla_{l_1} \nabla_{l_2} \cdots \nabla_{l_{m-1}} \nabla_k R_{ij} &= \nabla_{l_1} \nabla_{l_2} \cdots \nabla_{l_{m-1}} \nabla_k R_{ji} = -\nabla_{l_1} \nabla_{l_2} \cdots \nabla_{l_{m-1}} \nabla_i R_{jk} = -\nabla_{l_1} \nabla_{l_2} \cdots \nabla_{l_{m-1}} \nabla_i R_{kj} = \\ &= \nabla_{l_1} \nabla_{l_2} \cdots \nabla_{l_{m-1}} \nabla_j R_{ki} = \nabla_{l_1} \nabla_{l_2} \cdots \nabla_{l_{m-1}} \nabla_j R_{ik} = -\nabla_{l_1} \nabla_{l_2} \cdots \nabla_{l_{m-1}} \nabla_k R_{ij}. \end{aligned}$$

Comparig the first and the last formulas, we obtain: $\nabla_{l_1} \nabla_{l_2} \cdots \nabla_{l_{m-1}} \nabla_{l_m} R_{ij} = 0$.

In general case, it does not have to be true in the spaces with affine connections. The concept of generalized Ricci-symmetric spaces was first introduced by Berezovski et al. [33–35].

3. Basic Concepts of the Theory of Geodesic Mappings of Spaces with Affine Connections

Let us suppose that a space A_n with an affine connection admits a one-to-one differentiable mapping f onto another space $\bar{A}_n$ with an affine connection, and the inverse mapping is differentiable too. Locally, the spaces are referred to as a common coordinate system $x = (x^1, x^2, \ldots, x^n)$.

Assume the deformation tensor of the respective mapping f has the form

$$P^h_{ij}(x) = \bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x), \tag{1}$$

where $\Gamma^h_{ij}(x)$ and $\bar{\Gamma}^h_{ij}(x)$ are components of affine connections without torsion of the spaces A_n and $\bar{A}_n$, respectively.

A diffeomorphism $f : A_n \to \bar{A}_n$ is called a *geodesic* mapping if any geodesic on A_n is mapped under f onto a geodesic on $\bar{A}_n$.

It is known [1–13,43] that in order for a mapping f of a space with an affine connection A_n onto another space with an affine connection $\bar{A}_n$ to be geodesic, it is necessary and sufficient that in a common coordinate system the deformation tensor can be represented in the form

$$P_{ij}^h(x) = \psi_i(x)\delta_j^h + \psi_j(x)\delta_i^h, \tag{2}$$

where δ_i^h is the Kronecker delta and ψ_i is a covector. A geodesic mapping is called *non-trivial* if $\psi_i \not\equiv 0$. If $\psi_i \equiv 0$, then f is a *trivial* geodesic of an *affine*.

It is obvious that any space A_n with an affine connection admits a (non-trivial) geodesic mapping onto a space $\bar{A}_n$ with an affine connection. In a given space, we can arbitrarily choose a covector field ψ, and with the help of Formula (2), there is the space $\bar{A}_n$—which is geodesicaly equivalent to the space A_n. We note that if A_n and $\bar{A}_n$ are equiaffine spaces (i.e., the Ricci tensors of A_n and $\bar{A}_n$ are symmetric; see [11–13,43]), then ψ_i is gradient-like vector field. This is true when, for example, A_n and $\bar{A}_n$ are (pseudo-)Riemannian.

Sinyukov opened the class of problems when the given space admits geodesic mappings onto other spaces. He [16] proved that equaffine symmetric and recurrent spaces do not admit non-trivial geodesic mappings onto (pseudo-)Riemannian spaces with non-constant curvatures. Mikeš [18–21] proved a very similar problem for equiaffine m-recurrent and also (Ricci) m-symmetric spaces; see Sinyukov [9] and Mikeš et al. [11–13,18–21]. In these cases, $\psi_i \equiv 0$.

Obviously, the right-hand side of Formula (2) depends on n arbitrary functions, which are generally arbitrary. In the case of the natural requirements for spaces A_n and $\bar{A}_n$, the fundamental conditions of geodesic mappings in the form of a closed system of differential equations of the Cauchy type were found in the covariant derivatives. For these systems, there exist regular solution methods; see [11–13]. A natural question is: *Under what conditions can the main equations for the mappings be obtained such a system*?

Sinyukov (see [9–13]) proved that the main equations for geodesic mappings between (pseudo-)Riemannian spaces V_n and $\bar{V}_n$ are equivalent to some linear system of differential equations of the Cauchy type in covariant derivatives.

A similar result was obtained by Mikeš and Berezovski [51] for geodesic mappings of equiaffine spaces A_n onto (pseudo-)Riemannian spaces $\bar{V}_n$. This result holds not only for equiaffine spaces but also for general ones. It follows from the fact that J.M. Thomas [4] proved that any space with an affine connection is projectively equivalent to an equiaffine space. This solves the problem of the projective metrizability of spaces A_n.

Interesting assumptions for spaces admitting geodesic and almost geodesic mappings are found in [32–35]. Our work presents even more general assumptions that lead to the fundamental equations of geodesic mappings having the Cauchy form.

It is known [9,12,13] that in a common coordinate system x, respective to the mapping, the components of the Riemannian tensors R_{ijk}^h and $\bar{R}_{ijk}^h$ of spaces with affine connections A_n and $\bar{A}_n$, respectively, are in the relation

$$\bar{R}_{ijk}^h = R_{ijk}^h + P_{ik,j}^h - P_{ij,k}^h + P_{ik}^\alpha P_{j\alpha}^h - P_{ij}^\alpha P_{k\alpha}^h, \tag{3}$$

where the comma "," denotes the covariant derivative with respect to the connection of the A_n.

On the basis of Forumals (2) and (3), we have

$$\bar{R}_{ijk}^h = R_{ijk}^h - \delta_j^h \psi_{i,k} + \delta_k^h \psi_{i,\,j} - \delta_i^h \psi_{j,k} + \delta_i^h \psi_{k,\,j} + \delta_j^h \psi_i \psi_k - \delta_k^h \psi_i \psi_j. \tag{4}$$

Contracting Formula (4) for h and k, we obtain

$$\bar{R}_{ij} = R_{ij} + n\psi_{i,j} - \psi_{j,i} + (1-n)\psi_i\psi_j, \tag{5}$$

where R_{ij} and $\bar{R}_{ij}$ are the Ricci tensors of the spaces with affine connections A_n and $\bar{A}_n$, respectively.

From Formula (5), we obtain

$$\psi_{i,j} = \frac{2}{n^2 - 1}\left[n\bar{R}_{ij} + \bar{R}_{ji} - (nR_{ij} + R_{ji})\right] + \psi_i\psi_j. \tag{6}$$

Since

$$\bar{R}^h_{ijk;m} = \frac{\partial \bar{R}^h_{ijk}}{\partial x^m} + \bar{\Gamma}^h_{m\alpha}\bar{R}^\alpha_{ijk} - \bar{\Gamma}^\alpha_{mi}\bar{R}^h_{\alpha jk} - \bar{\Gamma}^\alpha_{mj}\bar{R}^h_{i\alpha k} - \bar{\Gamma}^\alpha_{mk}\bar{R}^h_{ij\alpha},$$

from Formula (1), we have

$$\bar{R}^h_{ijk;m} = \bar{R}^h_{ijk,m} + P^h_{m\alpha}\bar{R}^\alpha_{ijk} - P^\alpha_{mi}\bar{R}^h_{\alpha jk} - P^\alpha_{mj}\bar{R}^h_{i\alpha k} - P^\alpha_{mk}\bar{R}^h_{ij\alpha}. \tag{7}$$

From now on, the semicolon " ; " denotes the covariant derivative with respect to the connection of the space $\bar{A}_n$.

Contracting Formula (7) for h and k, we obtain

$$\bar{R}_{ij;m} = \bar{R}_{ij,m} - P^\alpha_{mi}\bar{R}_{\alpha j} - P^\alpha_{mj}\bar{R}_{i\alpha}. \tag{8}$$

Symmetrizing Formula (8) in the indices i and m, we obtain

$$\bar{R}_{ij;m} + \bar{R}_{mj;i} = \bar{R}_{ij,m} + \bar{R}_{mj,i} - 2P^\alpha_{mi}\bar{R}_{\alpha j} - P^\alpha_{mj}\bar{R}_{i\alpha} - P^\alpha_{ij}\bar{R}_{m\alpha}. \tag{9}$$

Recalling that in the case of geodesic mappings of affinely connected spaces $A_n \to \bar{A}_n$, the deformation tensor P^k_{ij} is expressed by Formula (2), from Formula (9), we obtain

$$\bar{R}_{ij;m} + \bar{R}_{mj;i} = \bar{R}_{ij,m} + \bar{R}_{mj,i} - 3\psi_m\bar{R}_{ij} - 3\psi_i\bar{R}_{mj} - \psi_j(\bar{R}_{im} + \bar{R}_{mi}). \tag{10}$$

4. Geodesic Mappings of Spaces with Affine Connections onto Generalized 2-Ricci-Symmetric Spaces

A space $\bar{A}_n$ with an affine connection is called *generalized Ricci-symmetric* if the Ricci tensor $\bar{R}_{ij}$ for the space satisfies the conditions

$$\bar{R}_{ij;k} + \bar{R}_{kj;i} = 0.$$

A space $\bar{A}_n$ with an affine connection is called *generalized* 2*-Ricci-symmetric* if its Ricci tensor $\bar{R}_{ij}$ satisfies the conditions

$$\bar{R}_{ij;km} + \bar{R}_{kj;im} = 0. \tag{11}$$

Obviously, generalized Ricci-symmetric spaces are generalized 2-Ricci-symmetric spaces. By means of Formula (1), we have

$$\begin{aligned} \left(\bar{R}_{im;j}\right)_{,k} &= \bar{R}_{im;jk} + \bar{R}_{\alpha m;j}P^\alpha_{ik} + \bar{R}_{i\alpha;j}P^\alpha_{mk} + \bar{R}_{im;\alpha}P^\alpha_{jk}, \\ \bar{R}_{ij;k} &= \bar{R}_{ij,k} - \bar{R}_{i\alpha}P^\alpha_{jk} - \bar{R}_{\alpha j}P^\alpha_{ik}. \end{aligned} \tag{12}$$

By covariant differentiation for Formula (10), with respect to the connection of the space A_n, expressing the left-hand side of Formula (12) and taking account of Formula (2), we find

$$\begin{aligned} \bar{R}_{ij;mk} + \bar{R}_{mj;ik} &= \bar{R}_{ij,mk} + \bar{R}_{mj,ik} - 3\psi_{m,k}\bar{R}_{ij} - 3\psi_{i,k}\bar{R}_{mj} - \psi_{j,k}\left(\bar{R}_{im} + \bar{R}_{mi}\right) \\ &-\left(3\bar{R}_{ij,k} + \bar{R}_{ik,j} + \bar{R}_{ki,j}\right)\psi_m - \left(3\bar{R}_{mj,k} + \bar{R}_{km,j} + \bar{R}_{mk,j}\right)\psi_i - 2\left(\bar{R}_{im,k} + \bar{R}_{mi,k}\right)\psi_j \\ -3\left(\bar{R}_{im,j} + \bar{R}_{mi,j}\right)\psi_k &+ 8\left(\bar{R}_{im} + \bar{R}_{mi}\right)\psi_j\psi_k + 4\left(\bar{R}_{jm} + \bar{R}_{mj}\right)\psi_i\psi_k + 4\left(\bar{R}_{ij} + \bar{R}_{ji}\right)\psi_k\psi_m \\ &+3\left(\bar{R}_{mk} + \bar{R}_{km}\right)\psi_i\psi_j + 2\left(\bar{R}_{kj} + \bar{R}_{jk}\right)\psi_i\psi_m + 3\left(\bar{R}_{ik} + \bar{R}_{ki}\right)\psi_j\psi_m. \end{aligned} \tag{13}$$

By means of Formula (6), Formula (13) is expressible in the form

$$\begin{aligned}\bar{R}_{ij;mk}+\bar{R}_{mj;ik}=\bar{R}_{ij,mk}+\bar{R}_{mj,ik}-\left(3\bar{R}_{ij,k}+\bar{R}_{ik,j}+\bar{R}_{ki,j}\right)\psi_m\\ -\left(3\bar{R}_{mj,k}+\bar{R}_{km,j}+\bar{R}_{mk,j}\right)\psi_i-2\left(\bar{R}_{im,k}+\bar{R}_{mi,k}\right)\psi_j-3\left(\bar{R}_{im,j}+\bar{R}_{mi,j}\right)\psi_k-T_{ijmk},\end{aligned} \tag{14}$$

where

$$\begin{aligned}T_{ijmk}=3\left[\frac{1}{n^2-1}\left(n\bar{R}_{mk}+\bar{R}_{km}-\left(nR_{mk}+R_{km}\right)\right)+\psi_m\psi_k\right]\bar{R}_{ij}\\ +3\left[\frac{1}{n^2-1}\left(n\bar{R}_{ik}+\bar{R}_{ki}-\left(nR_{ik}+R_{ki}\right)\right)+\psi_i\psi_k\right]\bar{R}_{mj}\\ +\left[\left(\frac{1}{n^2-1}\left(n\bar{R}_{jk}+\bar{R}_{kj}-\left(nR_{jk}+R_{kj}\right)\right)+\psi_j\psi_k\right]\left(\bar{R}_{im}+\bar{R}_{mi}\right)\\ -8\left(\bar{R}_{im}+\bar{R}_{mi}\right)\psi_j\psi_k-4\left(\bar{R}_{jm}+\bar{R}_{mj}\right)\psi_i\psi_k-4\left(\bar{R}_{ij}+\bar{R}_{ji}\right)\psi_k\psi_m\\ -3\left(\bar{R}_{mk}+\bar{R}_{km}\right)\psi_i\psi_j-2\left(\bar{R}_{kj}+\bar{R}_{jk}\right)\psi_i\psi_m-3\left(\bar{R}_{ik}+\bar{R}_{ki}\right)\psi_j\psi_m.\end{aligned} \tag{15}$$

Let us assume that the space $\bar{A}_n$ is generalized 2-Ricci-symmetric. Then, the Ricci tensor $\bar{R}_{ij}$ of the space $\bar{A}_n$ satisfies the conditions of Formula (11).

By means of Formula (11), it follows from Formula (14) that

$$\begin{aligned}\bar{R}_{ij,mk}+\bar{R}_{mj,ik}=\left(3\bar{R}_{ij,k}+\bar{R}_{ik,j}+\bar{R}_{ki,j}\right)\psi_m+\left(3\bar{R}_{mj,k}+\bar{R}_{km,j}+\bar{R}_{mk,j}\right)\psi_i\\ +2\left(\bar{R}_{im,k}+\bar{R}_{mi,k}\right)\psi_j+3\left(\bar{R}_{im,j}+\bar{R}_{mi,j}\right)\psi_k+T_{ijmk}.\end{aligned} \tag{16}$$

Let us alternate Formula (16) with respect to the indices i and k. Because of the Ricci identity, we obtain

$$\begin{aligned}\bar{R}_{ij,mk}-\bar{R}_{kj,mi}=\bar{R}_{\alpha j}R^{\alpha}_{mki}+\bar{R}_{m\alpha}R^{\alpha}_{jki}+3\left(\bar{R}_{ij,k}-\bar{R}_{kj,i}\right)\psi_m\\ +2\left(\bar{R}_{im,k}+\bar{R}_{mi,k}-\bar{R}_{km,i}-\bar{R}_{mk,i}\right)\psi_j+3\bar{R}_{mj,k}\psi_i-3\bar{R}_{mj,i}\psi_k+T_{ijmk}-T_{kjmi}.\end{aligned} \tag{17}$$

By means of the Ricci identity and the properties of a curvature tensor, Formula (13) is expressible in the form

$$\begin{aligned}\bar{R}_{ij,km}-\bar{R}_{kj,im}=2\bar{R}_{\alpha j}R^{\alpha}_{mki}+\bar{R}_{i\alpha}R^{\alpha}_{jkm}+\bar{R}_{k\alpha}R^{\alpha}_{jmi}+\bar{R}_{m\alpha}R^{\alpha}_{jki}+3\left(\bar{R}_{ij,k}-\bar{R}_{kj,i}\right)\psi_m\\ +2\left(\bar{R}_{im,k}+\bar{R}_{mi,k}-\bar{R}_{km,i}-\bar{R}_{mk,i}\right)\psi_j+3\bar{R}_{mj,k}\psi_i-3\bar{R}_{mj,i}\psi_k+T_{ijmk}-T_{kjmi}.\end{aligned} \tag{18}$$

Let us interchange k and m in Formula (18). Then, adding it to Formula (16), we have

$$\begin{aligned}2\bar{R}_{ij,mk}=&2\bar{R}_{\alpha j}R^{\alpha}_{kmi}+\bar{R}_{i\alpha}R^{\alpha}_{jkm}+\bar{R}_{m\alpha}R^{\alpha}_{jki}+\bar{R}_{k\alpha}R^{\alpha}_{jmi}\\ &+\left(3\bar{R}_{ij,k}-3\bar{R}_{kj,i}+\bar{R}_{ik,j}+\bar{R}_{ki,j}\right)\psi_m+\left(3\bar{R}_{mj,k}+3\bar{R}_{kj,m}+\bar{R}_{km,j}+\bar{R}_{mk,j}\right)\psi_i\\ &+2\left(\bar{R}_{im,k}+\bar{R}_{mi,k}+\bar{R}_{ik,m}+\bar{R}_{ki,m}-\bar{R}_{mk,i}-\bar{R}_{km,i}\right)\psi_j\\ &+3\left(\bar{R}_{im,j}+\bar{R}_{mi,j}+\bar{R}_{ij,m}-\bar{R}_{mj,i}\right)\psi_k+T_{ijmk}+T_{ijkm}-T_{mjki}.\end{aligned} \tag{19}$$

Let us introduce the tensor $\bar{R}_{ijk}$, defined by

$$\bar{R}_{ij,k}=\bar{R}_{ijk}. \tag{20}$$

By means of Formula (20), Formula (19) is expressible in the form

$$\begin{aligned}2\bar{R}_{ijm,k}=&2\bar{R}_{\alpha j}R^{\alpha}_{kmi}+\bar{R}_{i\alpha}R^{\alpha}_{jkm}+\bar{R}_{m\alpha}R^{\alpha}_{jki}+\bar{R}_{k\alpha}R^{\alpha}_{jmi}\\ &+\left(3\bar{R}_{ijk}-3\bar{R}_{kji}+\bar{R}_{ikj}+\bar{R}_{kij}\right)\psi_m+\left(3\bar{R}_{mjk}+3\bar{R}_{kjm}+\bar{R}_{kmj}+\bar{R}_{mkj}\right)\psi_i\\ &+2\left(\bar{R}_{imk}+\bar{R}_{mik}+\bar{R}_{ikm}+\bar{R}_{kim}-\bar{R}_{mki}-\bar{R}_{kmi}\right)\psi_j\\ &+3\left(\bar{R}_{imj}+\bar{R}_{mij}+\bar{R}_{ijm}-\bar{R}_{mji}\right)\psi_k+T_{ijmk}+T_{ijkm}-T_{mjki}.\end{aligned} \tag{21}$$

In the following, we have assumed that space A_n with an affine connection is given. Then, taking account of the structure of the tensor T_{ijlk}, which was determined by Formula (15), we see that the right-hand side of Equation (20) depends on unknown tensors ψ_i, $\bar{R}_{ij}$, $\bar{R}_{ijk}$.

Obviously, in the space A_n, Equations (6), (20), and (21) form a closed system of PDEs of the Cauchy type with respect to the functions $\psi_i(x)$, $\bar{R}_{ij}(x)$, and $\bar{R}_{ijk}(x)$.

We have proved the following theorem:

Theorem 1. *In order for a space A_n with an affine connection to admit geodesic mapping onto a generalized 2-Ricci-symmetric space $\bar{A}_n$, it is necessary and sufficient that the system of differential equations of the Cauchy type in the covariant derivatives of Formulas (6), (20), and (21) has a solution with respect to the unknown functions $\psi_i(x)$, $\bar{R}_{ij}(x)$, $\bar{R}_{ijk}(x)$.*

By the elementary sum of unknown function, we make sure that the following holds:

Consequence. The general solution of the mixed system of the Cauchy type including Formulas (6), (20), and (21) depends on no more than $n + \frac{1}{2}\left(n^2 + n^3\right)$ essential parameters.

5. Geodesic Mappings of Spaces with Affine Connections onto Generalized 3-Ricci-Symmetric Spaces

A space $\bar{A}_n$ with an affine connection is called *generalized 3-Ricci-symmetric* if the Ricci tensor $\bar{R}_{ij}$ for the space satisfies the conditions

$$\bar{R}_{ij;kml} + \bar{R}_{kj;iml} = 0. \tag{22}$$

Obviously, generalized 2-Ricci-symmetric spaces are generalized 3-Ricci-symmetric spaces.

Observe that Formulas (6) and (14) were obtained for the general case of geodesic mappings of spaces with affine connections.

Firstly, because of Formula (1), we obtain

$$\bar{R}_{ij;mk} = \bar{R}_{ij,mk} - \bar{R}_{i\alpha,k}P^{\alpha}_{jm} - \bar{R}_{i\alpha}P^{\alpha}_{jm,k} - \bar{R}_{\alpha j,k}P^{\alpha}_{im} - \bar{R}_{\alpha j}P^{\alpha}_{im,k} - P^{\alpha}_{ik}\bar{R}_{\alpha j;m} - P^{\alpha}_{jk}\bar{R}_{i\alpha;m} - P^{\alpha}_{mk}\bar{R}_{ij;\alpha},$$

$$\left(\bar{R}_{ij;mk}\right)_{,l} = \bar{R}_{ij;mkl} + P^{\alpha}_{il}\bar{R}_{\alpha j;mk} + P^{\alpha}_{jl}\bar{R}_{i\alpha;mk} + P^{\alpha}_{ml}\bar{R}_{ij;\alpha k} + P^{\alpha}_{kl}\bar{R}_{ij;m\alpha}.$$

From these equations, taking account of Formulas (2) and (12), we find

$$\begin{aligned} \bar{R}_{ij;mk} &= \Omega_{ijmk}, \\ \left(\bar{R}_{ij;mk}\right)_{,l} &= \bar{R}_{ij;mkl} + 4\psi_l\Omega_{ijmk} + \psi_i\Omega_{ljmk} + \psi_j\Omega_{ilmk} + \psi_m\Omega_{ijlk} + \psi_k\Omega_{ijml}, \end{aligned} \tag{23}$$

where

$$\begin{aligned} \Omega_{ijmk} = \bar{R}_{ij,mk} - \psi_j\bar{R}_{im,k} - 3\psi_m\bar{R}_{ij,k} - \psi_i\bar{R}_{mj,k} - \psi_i\bar{R}_{kj,m} - 3\psi_k\bar{R}_{ij,m} - \psi_j\bar{R}_{ik,m} \\ +4\psi_k\psi_i\bar{R}_{mj} + 3\psi_i\psi_m\bar{R}_{kj} + \psi_i\psi_j\left(\bar{R}_{km} + \bar{R}_{mk}\right) + 8\psi_k\psi_m\bar{R}_{ij} \\ +4\psi_k\psi_j\bar{R}_{im} + 3\psi_m\psi_j\bar{R}_{ik} - 2\psi_{m,k}\bar{R}_{ij} - \psi_{j,k}\bar{R}_{im} - \psi_{i,k}\bar{R}_{mj}. \end{aligned} \tag{24}$$

Let us assume that in Formula (24), the covariant derivatives of the vector ψ_i with respect to the connection of the space A_n are expressed according to Formula (6).

Let us covariantly differentiate Formula (14) with respect to the connection of the space A_n. Then, using Formula (23), expressing, on the left-hand side, the third covariant derivatives of $\bar{R}_{ij}$ with respect to the connection of A_n, in terms of the third covariant derivatives with respect to the connection of the space $\bar{A}_n$, and transforming the formula, we obtain

$$\bar{R}_{ij;mkl} + \bar{R}_{mj;ikl} = \bar{R}_{ij,mkl} + \bar{R}_{mj,ikl} - \Omega_{ijmkl}, \tag{25}$$

where

$$\begin{aligned}\Omega_{ijmkl} = {} & (3\bar{R}_{ij,kl} + \bar{R}_{ik,jl} + \bar{R}_{ki,jl})\psi_m + (3\bar{R}_{ij,k} + \bar{R}_{ik,j} + \bar{R}_{ki,j})\psi_{m,l} \\ & + (3\bar{R}_{mj,kl} + \bar{R}_{km,jl} + \bar{R}_{mk,jl})\psi_i + (3\bar{R}_{mj,k} + \bar{R}_{km,j} + \bar{R}_{mk,j})\psi_{i,l} \\ & + 2(\bar{R}_{im,kl} + \bar{R}_{mi,kl})\psi_j + 2(\bar{R}_{im,k} + \bar{R}_{mi,k})\psi_{j,l} + 3(\bar{R}_{im,jl} + \bar{R}_{mi,jl})\psi_k \\ & + 3(\bar{R}_{im,j} + \bar{R}_{mi,j})\psi_{k,l} + T_{ijmk,l} + 4\psi_l(\Omega_{ijmk} + \Omega_{mjik}) \\ & + \psi_i(\Omega_{ljmk} + \Omega_{mjlk}) + \psi_j(\Omega_{ilmk} + \Omega_{mlik}) \\ & + \psi_m(\Omega_{ijkl} + \Omega_{ljki}) + \psi_k(\Omega_{ijml} + \Omega_{mjil}).\end{aligned} \tag{26}$$

Again, assume that in Formula (26) the covariant derivatives of the vector ψ_i with respect to the connection of the space A_n are expressed according to Formula (6).

Suppose that the space $\bar{A}_n$ is generalized 3-Ricci-symmetric. Then, the Ricci tensor $\bar{R}_{ij}$ of the space $\bar{A}_n$ satisfies the conditions Formula (22).

Taking account of Formula (22), from Formula (25), it follows that

$$\bar{R}_{ij,mkl} + \bar{R}_{mj,ikl} = \Omega_{ijmkl}. \tag{27}$$

Alternating Formula (27) with respect to the indices i and k, and taking account of the Ricci identity, we obtain

$$\bar{R}_{ij,mkl} - \bar{R}_{kj,mil} = \Omega_{ijmkl} - \Omega_{kjmil} + \bar{R}_{\alpha j,l}R^{\alpha}_{mki} + \bar{R}_{\alpha j}R^{\alpha}_{mki,l} + \bar{R}_{m\alpha,l}R^{\alpha}_{jki} + \bar{R}_{m\alpha}R^{\alpha}_{jki,l}. \tag{28}$$

Because of the Ricci identity and the properties of a curvature tensor, Formula (28) is expressible in the form

$$\begin{aligned}\bar{R}_{ij,kml} - \bar{R}_{kj,iml} = {} & \Omega_{ijmkl} - \Omega_{kjmil} + 2\bar{R}_{\alpha j,l}R^{\alpha}_{mki} + 2\bar{R}_{\alpha j}R^{\alpha}_{mki,l} + \bar{R}_{i\alpha,l}R^{\alpha}_{jkm} \\ & + \bar{R}_{i\alpha}R^{\alpha}_{jkm,l} + \bar{R}_{k\alpha,l}R^{\alpha}_{jmi} + \bar{R}_{k\alpha}R^{\alpha}_{jmi,l} + \bar{R}_{m\alpha,l}R^{\alpha}_{jki} + \bar{R}_{m\alpha}R^{\alpha}_{jki,l}.\end{aligned} \tag{29}$$

Interchanging k and m in Formula (29) and adding it to Formula (27), we have

$$\begin{aligned}2\bar{R}_{ij,mkl} = {} & \Omega_{ijmkl} + \Omega_{ijkml} - \Omega_{mjkil} + 2\bar{R}_{\alpha j,l}R^{\alpha}_{kmi} + 2\bar{R}_{\alpha j}R^{\alpha}_{kmi,l} + \bar{R}_{i\alpha,l}R^{\alpha}_{jmk} \\ & + \bar{R}_{i\alpha}R^{\alpha}_{jmk,l} + \bar{R}_{m\alpha,l}R^{\alpha}_{jki} + \bar{R}_{m\alpha}R^{\alpha}_{jki,l} + \bar{R}_{k\alpha,l}R^{\alpha}_{jmi} + \bar{R}_{k\alpha}R^{\alpha}_{jmi,l}.\end{aligned} \tag{30}$$

Let us introduce the tensor $\bar{R}_{ijkm}$ defined by

$$\bar{R}_{ijk,m} = \bar{R}_{ijkm}. \tag{31}$$

By means of Formulas (20) and (31), Formula (30) may be written in the form

$$\begin{aligned}2\bar{R}_{ijmk,l} = {} & \Omega_{ijmkl} + \Omega_{ijkml} - \Omega_{mjkil} + 2\bar{R}_{\alpha jl}R^{\alpha}_{kmi} + 2\bar{R}_{\alpha j}R^{\alpha}_{kmi,l} + \bar{R}_{i\alpha l}R^{\alpha}_{jmk} \\ & + \bar{R}_{i\alpha}R^{\alpha}_{jmk,l} + \bar{R}_{m\alpha l}R^{\alpha}_{jki} + \bar{R}_{m\alpha}R^{\alpha}_{jki,l} + \bar{R}_{k\alpha l}R^{\alpha}_{jmi} + \bar{R}_{k\alpha}R^{\alpha}_{jmi,l}.\end{aligned} \tag{32}$$

In the following, we have assumed that a space A_n with affine connection is given. Then, the right-hand side of Formula (31) depends on ψ_i, $\bar{R}_{ij}$, $\bar{R}_{ijk}$, and $\bar{R}_{ijkm}$.

Obviously, in the space A_n Formulas (6), (20), (31), and (32) form a closed system of differential equations of the Cauchy type in covariant derivatives with respect to the functions $\psi_i(x)$, $\bar{R}_{ij}(x)$, $\bar{R}_{ijk}(x)$, and $\bar{R}_{ijkm}(x)$.

We obtain the following theorem:

Theorem 2. *In order for a space A_n with an affine connection to admit a geodesic mapping onto a generalized 3-Ricci-symmetric space $\bar{A}_n$, it is necessary and sufficient that the system of differential equations of the Cauchy type in covariant derivatives of Formulas (6), (20), (31), and (32) has a solution with respect to the unknown functions $\psi_i(x)$, $\bar{R}_{ij}(x)$, $\bar{R}_{ijk}(x)$, and $\bar{R}_{ijkm}(x)$.*

Consequence . The general solution of the mixed system of the Cauchy type including Formulas (6), (20), (31), and (32) depends on no more than $n + \frac{1}{2}(n^2 + n^4)$ essential parameters.

6. Geodesic Mappings of Spaces with Affine Connections onto Generalized m-Ricci-Symmetric Spaces

A space $\bar{A}_n$ with an affine connection is called *generalized m-Ricci-symmetric* if the Ricci tensor $\bar{R}_{ij}$ for the space satisfies the condition

$$\bar{R}_{ij;\rho_1\rho_2\ldots\rho_m} + \bar{R}_{\rho_1 j;i\rho_2\ldots\rho_m} = 0. \tag{33}$$

It is obvious that generalized 2-Ricci-symmetric spaces and generalized 3-Ricci-symmetric spaces are special cases of generalized m-Ricci-symmetric spaces with $m = 2$ and $m = 3$, respectively.

Let us covariantly differentiate $(m-3)$ times Formula (26) with respect to the connection of the space A_n. Then, expressing, in the left-hand side, the covariant derivatives with respect to the connection of the space A_n, in terms of the covariant derivatives with respect to the connection of the space $\bar{A}_n$, using the formula (Kaigorodov [29,30])

$$\begin{aligned}\left(\bar{R}_{ij;\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}}\right)_{,\rho_\tau} = \bar{R}_{ij;\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}\rho_\tau} + P^\alpha_{i\rho_\tau}\bar{R}_{\alpha j;\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}} \\ + P^\alpha_{j\rho_\tau}\bar{R}_{i\alpha;\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}} + P^\alpha_{\rho_1\rho_\tau}\bar{R}_{ij;\alpha\ldots\rho_{\tau-2}\rho_{\tau-1}} + \cdots + P^\alpha_{\rho_{\tau-1}\rho_\tau}\bar{R}_{ij;\rho_1\ldots\rho_{\tau-2}\alpha}.\end{aligned}$$

Transforming the left-hand side of Formula (33), we obtain

$$\bar{R}_{ij;\rho_1\rho_2\ldots\rho_m} + \bar{R}_{\rho_1 j;i\rho_2\ldots\rho_m} = \bar{R}_{ij,\rho_1\rho_2\ldots\rho_m} + \bar{R}_{\rho_1 j,i\rho_2\ldots\rho_m} - \Omega_{ij\rho_1\rho_2\ldots\rho_m}, \tag{34}$$

where the tensor $\Omega_{ij\rho_1\rho_2\ldots\rho_m}$ depends on unknown tensors ψ_i, $\bar{R}^h_{ij}$, $\bar{R}_{ij,\rho_1}$, and $\bar{R}_{ij,\rho_1\rho_2}, \ldots, \bar{R}_{ij,\rho_1\rho_2\ldots\rho_{m-1}}$. Observe that the tensor also depends on the known tensors, which are defined in the space A_n.

Suppose that the space $\bar{A}_n$ is generalized m-Ricci-symmetric. Then, the Ricci tensor $\bar{R}_{ij}$ of the space $\bar{A}_n$ satisfies the conditions of Formula (33).

By means of Formula (33), Formula (34) may be written in the form

$$\bar{R}_{ij,\rho_1\rho_2\rho_3\ldots\rho_m} + \bar{R}_{\rho_1 j,i\rho_2\rho_3\ldots\rho_m} = \Omega_{ij\rho_1\rho_2\rho_3\ldots\rho_m}, \tag{35}$$

Alternating Formula (35) with respect to the indices i and ρ_2, taking account of the Ricci identity, we have

$$\bar{R}_{ij,\rho_1\rho_2\rho_3\ldots\rho_m} - \bar{R}_{\rho_2 j,\rho_1 i\rho_3\ldots\rho_m} = \overset{1}{\Omega}_{ij\rho_1\rho_2\rho_3\ldots\rho_m}, \tag{36}$$

where

$$\overset{1}{\Omega}_{ij\rho_1\rho_2\rho_3\ldots\rho_m} = \Omega_{ij\rho_1\rho_2\rho_3\ldots\rho_m} - \Omega_{\rho_2 j\rho_1 i\rho_3\ldots\rho_m} - \left(\bar{R}_{\alpha j}R^\alpha_{\rho_1 i\rho_2} + \bar{R}_{\rho_1\alpha}R^\alpha_{ji\rho_2}\right)_{,\rho_3\ldots\rho_m}.$$

By means of the Ricci identity, Formula (36) is expressible in the form

$$\bar{R}_{ij,\rho_2\rho_1\rho_3\ldots\rho_m} - \bar{R}_{\rho_2 j,i\rho_1\rho_3\ldots\rho_m} = \overset{2}{\Omega}_{ij\rho_1\rho_2\rho_3\ldots\rho_m}, \tag{37}$$

where

$$\overset{2}{\Omega}_{ij\rho_1\rho_2\rho_3\ldots\rho_m} = \overset{1}{\Omega}_{ij\rho_1\rho_2\rho_3\ldots\rho_m} + \left(\bar{R}_{\alpha j}R^\alpha_{\rho_1\rho_2 i} + \bar{R}_{i\alpha}R^\alpha_{j\rho_2\rho_1} + \bar{R}_{\rho_2\alpha}R^\alpha_{j\rho_1 i}\right)_{,\rho_3\ldots\rho_m}.$$

Interchanging ρ_1 and ρ_2 in Formula (37) and adding it to Formula (37), we have

$$2\bar{R}_{ij,\rho_1\rho_2\rho_3\cdots\rho_m} = \overset{3}{\Omega}_{ij\rho_1\rho_2\rho_3\cdots\rho_m} \tag{38}$$

where

$$\overset{3}{\Omega}_{ij\rho_1\rho_2\rho_3\cdots\rho_m} = \Omega_{ij\rho_1\rho_2\rho_3\cdots\rho_m} + \overset{2}{\Omega}_{ij\rho_2\rho_1\rho_3\cdots\rho_m} .$$

Taking account of the structure of the tensor $\overset{3}{\Omega}_{ij\rho_1\rho_2\rho_3\cdots\rho_m}$, we see that the right-hand side of Formula (38) depends on unknown tensors ψ_i, $\bar{R}_{ij}$, $\bar{R}_{ij,\rho_1}$, $\bar{R}_{ij,\rho_1}$, $\bar{R}_{ij,\rho_1\rho_2}$, $\ldots$, $\bar{R}_{ij,\rho_1\rho_2\cdots\rho_{m-1}}$. The tensor also depends on the known tensors, which are defined in the space A_n. Let us introduce the tensors $\bar{R}^h_{ij\rho_1\rho_2\rho_3}, \ldots, \bar{R}^h_{ij\rho_1\rho_2\rho_3\cdots\rho_{m-2}\rho_{m-1}}$, defined by

$$\begin{gathered} \bar{R}^h_{ij\rho_1\rho_2,\rho_3} = \bar{R}^h_{ij\rho_1\rho_2\rho_3}, \\ \cdots\cdots\cdots\cdots \\ \bar{R}^h_{ij\rho_1\rho_2\rho_3\cdots\rho_{m-2},\rho_{m-1}} = \bar{R}^h_{ij\rho_1\rho_2\rho_3\cdots\rho_{m-2}\rho_{m-1}}. \end{gathered} \tag{39}$$

Because of Formulas (20), (31) and (39), Formula (38) is expressible in the form

$$2\bar{R}_{ij\rho_1\rho_2\rho_3\cdots\rho_{m-1},\rho_m} = \overset{3}{\Omega}_{ij\rho_1\rho_2\rho_3\cdots\rho_{m-1}\rho_m} \tag{40}$$

where the tensor $\overset{3}{\Omega}_{ij\rho_1\rho_2\rho_3\cdots\rho_{m-1}\rho_m}$ depends on unknown tensors ψ_i, $\bar{R}_{ij}$, $\bar{R}_{ij\rho_1}$, $\bar{R}_{ij\rho_1\rho_2}$, $\ldots$, $\bar{R}_{ij\rho_1\rho_2\cdots\rho_{m-1}\rho_{m-1}}$, and also it depends on the known tensors, which are defined in the space A_n.

Let us assume that a space A_n with affine connection is given. Then, in the space A_n, the Equations (6), (20), (31), (39) and (40) form a closed system of differential equations of Cauchy type in covariant derivatives with respect to the functions $\psi_i(x)$, $\bar{R}_{ij}(x)$, $\bar{R}_{ij\rho_1}(x)$, $\bar{R}_{ij\rho_1\rho_2}(x)$, $\ldots$, $\bar{R}_{ij\rho_1\rho_2\cdots\rho_{m-2}\rho_{m-1}}(x)$.

We obtain the following theorem.

Theorem 3. *In order that space A_n with affine connection admits geodesic mapping onto a generalized m-Ricci-symmetric space $\bar{A}_n$, it is necessary and sufficient that the system of differential equations of Cauchy type in covariant derivatives* (6), (20), (31), (39) *and* (40) *has a solution with respect to the unknown functions $\psi_i(x)$, $\bar{R}_{ij}(x)$, $\bar{R}_{ij\rho_1}(x)$, $\bar{R}_{ij\rho_1\rho_2}(x)$, $\ldots$, $\bar{R}_{ij\rho_1\rho_2\cdots\rho_{m-2}\rho_{m-1}}(x)$.*

Consequence. The general solution of the mixed system of the Cauchy type including Formulas (6), (20), (31), (39), and (40) depends on no more than $n + \frac{1}{2}\left(n^2 + n^{m+1}\right)$ essential parameters.

7. Conclusions

In this paper, we study geodesic mappings of spaces with affine connections onto generalized m-Ricci-symmetric spaces. For these cases, we obtain fundamental equations in the form of the system of differential equations of the Cauchy type in covariant derivatives; the general solutions depend on the real parameters.

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