International Journal of Pure and Applied Mathematics

Volume 45 No. 3 2008, 413-418

GEODESIC MAPPINGS OF WEAKLY BERWALD SPACES AND BERWALD SPACES ONTO RIEMANNIAN SPACES

Josef Mikeš[1 §], Sándor Bácsó[2], Vladimir Berezovski[3]

[1]Department of Algebra and Geometry
Faculty of Science
Palacky University
Tomkova 40, Olomouc, 771 47, CZECH REPUBLIC
e-mail: josef.mikes@upol.cz

[2]Faculty of Informatics
Debrecen University
Debrecen, P.O. Box 12, HUNGARY
e-mail: bacsos@inf.unideb.hu

[3]University of Uman
Institutskaya 1, Uman, UKRAINE
e-mail: berez.volod@rambler.ru

Abstract: The present paper is devoted to the investigation of the geodesic mappings of weakly Berwald spaces and Berwald spaces which are special Finsler spaces F_n, onto the Riemannian spaces $\bar{V}_n$. We can see that not every Finsler space admits non-trivial geodesic mapping onto a Riemannian space.

AMS Subject Classification: 53B20, 53B30, 53B40
Key Words: geodesic mapping, equations of Cauchy type, Finsler spaces, Berwald spaces, weakly Berwald spaces, Riemannian spaces

1. Geodesic Mappings of Finsler Spaces

It is known [4, 6, 9, 10], a *Finsler space* F_n, determined by the symmetric and regular metric tensor $g_{ij}(x^1, x^2, \dots, x^n, y^1, y^2, \dots, y^n)$, determined by a function $F(x, y) \equiv \frac{1}{2} L^2(x, y)$:

Received: February 11, 2008

§Correspondence author

$$g_{ij}(x,y) = \frac{\partial^2 F(x,y)}{\partial y^i \partial y^j}.$$

The *Berwald connection* of F_n is introduced by the formulas:

$$G^h(x,y) = \frac{1}{2}\, g^{ij} \left(\frac{\partial^2 F(x,y)}{\partial y^j \partial x^k} y^k - \frac{\partial F(x,y)}{\partial y^j} \right),$$

$$G^h_i(x,y) = \frac{\partial G^h(x,y)}{\partial y^i}, \qquad G^h_{ij}(x,y) = \frac{\partial^2 G^h(x,y)}{\partial y^i \partial y^j},$$

where g^{ij} are components of the matrix inverted to $\|g_{ij}(x,y)\|$.

For metric tensor of F_n we have the following formula:

$$g_{ij;k} = -2P_{ijk},$$

where $P_{ijk} = C_{ijk;\alpha} y^\alpha$ is the Landsberg tensor, and $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ is the Cartan tensor. Hereafter the semicolon ";" denotes the *covariant derivative* with respect to the Berwald connection of the space F_n.

We consider two Finsler spaces $F_n = (M_n, L)$ and $\bar{F}_n = (M_n, \bar{L})$ on the same underlying manifold M_n. If any geodesic of a Finsler space F_n coincides with a geodesic of $\bar{F}_n$ as a set of points and vice versa, the change $L \to \bar{L}$ of the metrics is called *projective*, and the mapping $F_n \to \bar{F}_n$ is a *geodesic mapping* between F_n and $\bar{F}_n$.

It is known that if a Finsler space F_n admits a geodesic mapping onto a Finsler space $\bar{F}_n$ then in general with respect to the mapping system of coordinates $x^1, x^2, \dots, x^n, y^1, y^2, \dots, y^n$ the objects of the Berwald connection of these spaces $G^h(x,y)$ and $\bar{G}^h(x,y)$ have the following relation [1, 2, 3, 5, 6, 9, 10]:

$$\bar{G}^h(x,y) = G^h(x,y) + \psi(x,y)\, y^h\,, \tag{1}$$

where $\psi(x,y)$ is a positively homogeneous in y of degree one function.

The formula (1) equvivalent to next formulas

$$\bar{G}^h_i(x,y) = G^h_i(x,y) + \delta^h_i \psi(x,y) + \psi_i(x,y)\, y^h,$$

$$\bar{G}^h_{ij}(x,y) = G^h_{ij}(x,y) + \delta^h_i \psi_j + \delta^h_j \psi_i + \psi_{ij}(x,y)\, y^h,$$

where

$$\psi_i = \frac{\partial \psi(x,y)}{\partial y^i}, \quad \psi_{ij} = \frac{\partial \psi_i(x,y)}{\partial y^j},$$

δ^h_i is the Kronecker symbol.

If $\psi_i \not\equiv 0$, then a geodesic mapping is called *nontrivial*; otherwise it is said to be *trivial*.

From the above mentioned equations we obtain

$$\bar{G}^h_{ijk}(x,y) = G^h_{ijk}(x,y) + \delta^h_i \psi_{jk} + \delta^h_j \psi_{ik} + \delta^h_k \psi_{ij} + \psi_{ijk}(x,y)\, y^h,$$

where

$$\psi_{ijk} = \frac{\partial \psi_{ij}(x,y)}{\partial y^k}.$$

Finsler space is a *Berwald type* if $G^h_{ijk} = 0$, and Finsler space is a *weakly Berwald type* if $G^\alpha_{ij\alpha} = G_{ij} = 0$.

Hence $\bar{G}_{ij} = G_{ij} + (n+1)\psi_{ij}$, so if we consider the geodesic mapping between Berwald and weakly Berwald spaces then $\psi_{ij} = 0$.

It is well known, that every positive definit Berwald metric has common geodesic with some Riemannian metric [12]. The aim of this paper to show a new method, which in suitable for the investigation of geodesic mapping of Berwald spaces onto Riemannian spaces in any cases.

2. Geodesic Mapping of Weakly Berwald Spaces onto Riemannian Spaces

We know, that the Douglas tensor is invariant under geodesic mapping, that is $D = \bar{D}$, where

$$D^h_{ijk} = G^h_{ijk} - \frac{1}{(n+1)}(G_{ijk}y^h + G_{ij}\delta^h_k + G_{ik}\delta^h_j + G_{jk}\delta^h_i), \text{ and } G_{ijk} = \frac{\partial G_{ij}}{\partial y^k}.$$

The Douglas tensor vanishes in Riemannian spaces, so in the weakly Berwald space we get $G^h_{ijk} = 0$, where the weakly Berwald space has common geodesics with a Riemannian space. So we obtain the following

Theorem 1. *If a weakly Berwald space has common geodesics with a Riemannian space, then the weakly Berwald space is a Berwald space.*

Berwald constructed a tensor, which is analogically of Weyl tensor of projective curvature. This tensor is also invariant under a geodesic mappings of Finsler spaces.

The fundamental equation (1) of geodesic mapping between Finsler spaces F_n and $\bar{F}_n$ equvivalent to following formula:

$$\overline{g}_{ij;k} = 2\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{ik} + \overline{g}_{i\alpha} y^\alpha \psi_{jk} + \overline{g}_{j\alpha} y^\alpha \psi_{ik} - 2\bar{P}_{ijk}.$$

If case that space F_n and $\bar{F}_n$ are weakly Berwald spaces this formula have following simly form:

$$\bar{g}_{ij;k} = 2\psi_k\bar{g}_{ij} + \psi_i\bar{g}_{jk} + \psi_j\bar{g}_{ik} - 2\bar{P}_{ijk}. \tag{2}$$

In this case the functions ψ_i are independent of y, i.e. $\psi_i = \psi_i(x)$.

3. Geodesic Mappings of Berwald Spaces onto Riemannian Spaces

Theorem 2. *The Berwald space F_n admits a non-trivial geodesic mapping onto a Riemannian space $\bar{V}_n$ with the metric tensor $\bar{g}_{ij}(x)$ if and only if the following system of differential equations with covariant derivatives of Cauchy type has a solution with respect to the symmetric tensor $\bar{g}_{ij}(x)$ ($\det\|\bar{g}_{ij}(x)\| \neq 0$), the non-zero vector $\psi_i(x)$ and the invariant $\mu(x)$:*

$$\begin{aligned}
&(a) & \bar{g}_{ij;k} &= 2\psi_k\bar{g}_{ij} + \psi_i\bar{g}_{jk} + \psi_j\bar{g}_{ik};\\
&(b) & n\psi_{i;j} &= n\psi_i\psi_j + \mu\bar{g}_{ij} - H_{ij} - \bar{g}_{i\alpha}\bar{g}^{\beta\gamma}H^{\alpha}_{\beta\gamma j} - \tfrac{2}{n+1}H^{\alpha}_{\alpha ij};\\
&(c) & (n-1)\mu_{;i} &= 2(n-1)\psi_\alpha\bar{g}^{\beta\gamma}H^{\alpha}_{\beta\gamma i}\\
& & &\quad +\psi_\alpha\bar{g}^{\alpha\beta}(5H_{\beta i} + \tfrac{6}{n+1}H^{\gamma}_{\gamma\beta i} - H_{i\beta})\\
& & &\quad +\bar{g}^{\alpha\beta}(H^{\gamma}_{\alpha\beta i;\gamma} - H_{\alpha i;\beta} - \tfrac{2}{n+1}H^{\gamma}_{\gamma\alpha i;\beta}),
\end{aligned} \tag{3}$$

where the semicolon denotes covariant derivative with respect to the space connection F_n, $\bar{g}^{ij}(x)$ are components of the matrix inverted to $\|\bar{g}_{ij}(x)\|$, H^h_{ijk} and H_{ij} are respectively curvature and Ricci tensors of the space F_n.

Proof. Suppose that F_n admits non trivial geodesic mapping onto $\bar{V}_n$ with metric tensor $\bar{g}_{ij}(x)$. Then the connections F_n and $\bar{V}_n$ have the relation (2) with $\bar{P}^h_{ijk} = 0$. If we consider the fundamental tensor $\bar{g}_{ij}(x)$ of $\bar{V}_n$, then we get the conditions which are sufficient for F_n to admit non trivial geodesic mapping onto $\bar{V}_n$.

Let us consider integrability conditions of the equations (3a)

$$\bar{g}_{h\alpha}H^{\alpha}_{ijk} + \bar{g}_{i\alpha}H^{\alpha}_{hjk} = 2\bar{g}_{hi}\psi_{[jk]} + \bar{g}_{j(h}\psi_{i)k} - \bar{g}_{k(h}\psi_{i)j}, \tag{4}$$

where $\psi_{ij} = \psi_{i;j} - \psi_i\psi_j$, $[i\,j]$ and $(i\,j)$ denotes alternation and symmetrization with respect to i and j, respective.

Transvecting (4) by $\bar{g}^{ij}$, we get $\psi_{[jk]} = \frac{1}{n+1}H^{\alpha}_{\alpha jk}$. Excluding $\psi_{[jk]}$ from (4), we obtain

$$\bar{g}_{\alpha(h}H^{\alpha}_{i)jk} - \frac{2}{n+1}\bar{g}_{hi}H^{\alpha}_{\alpha jk} = \bar{g}_{j(h}\psi_{i)k} - \bar{g}_{k(h}\psi_{i)j}. \qquad (5)$$

After the transvecting (5) by g^{ik} we easily obtain the conditions (3b) with $\mu \equiv \psi_{\alpha\beta}\bar{g}^{\alpha\beta}$.

If we consider the equation $\bar{g}^{ik}\bar{g}_{kj} = \delta^i_j$, it is not difficult to show that the equations (3a) are equivalent to the relations

$$\bar{g}^{\,ij}_{\;\;;k} = -2\psi_k\bar{g}^{ij} - \delta^i_k\psi^j - \delta^j_k\psi^i, \qquad (6)$$

where $\psi^i \equiv \psi_\alpha\bar{g}^{\alpha i}$.

We covariantly differentiate the conditions (3b) and then alternate the result with respect to the indices j and k taking into account (3a), (3b), (6) and transvecting by $\bar{g}^{ik}$, and finally we get equations (3c).

The theorem has been proved. □

From Theorem 2 we may conclude that the set of all Riemannian spaces $\bar{V}_n$, the given Berwald space F_n admits non trivial geodesic mapping onto $\bar{V}_n$, is dependent on $r \le r_0 = (n+1)(n+2)/2$ parameters.

Finding of all the solutions of (3) requires a consideration of their integrability conditions and differential extensions, which form a set of algebraic equations with respect to the unknown functions $\bar{g}_{ij}$, ψ_i and μ with coefficients from F_n. But this set is not linear and its solution is certainly difficult.

The results under discussion are generalisations of analogous theorems of N.S. Sinyukov [11] and J. Mikeš, V. Berezovski [7] for the geodesic mappings of Riemannian spaces.

Acknowledgements

Thsi work is supported by the Grant No. 201/05/2707 of The Czech Science Foundation and by the Council of Czech Government MSM No. 6198959214. The second author was supported by Hungary National Science Research Foundation OTKA No. T 48878.

References

[1] S. Bácsó, M. Matsumoto, Projective changes between Finsler spaces with (α,β)-metric, *Tensor N.S.*, **55** (1994), 252-257.

[2] S. Bácsó, M. Matsumoto, On Finsler spaces of Douglas type-A generalization of the notion of Berwald space, *Publ. Math. Debrecen*, **48** (1996), 357-366.

[3] S. Bácsó, I. Papp, A note on a generalized Douglas space, *Per. Math. Hung.*, **48**, No-s: 1-2 (2004), 181-184.

[4] D. Bao, S.S. Chern, Z. Shen, *Introduction to Riemann-Finsler Geometry*, Springer-Verlag (2000).

[5] M. Matsumoto, Projective change of Finsler metrics and projective flat Finsler spaces, *Tensor N.S.*, **34** (1980), 303-315.

[6] M. Matsumoto, Finsler geometry in the 20-th Century, *Handbook of Finsler Geometry* (Ed. P.L. Antonelli) (2005), 558-966.

[7] J. Mikeš, V. Berezovski, Geodesic mappings of affine-connected spaces onto Riemannian spaces, *Coll. Math. Soc.J. Bolyai. 56. Diff. Geometry, Eger.* (1989), 491-494, In Hungarian.

[8] J. Mikeš, Geodesic mappings of affine-connected and Riemannian spaces, *J. Math. Sci.*, New York, **78**, No. 3 (1996), 311-333.

[9] H. Rund, *The Differential Geometry of Finsler Spaces*, Springer-Verlag, Berlin-Göttingen-Heidelberg (1959).

[10] Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers (2001).

[11] N.S. Sinyukov, *Geodesic Mappings of Riemannian Spaces*, Moscow, Nauka (1979).

[12] Z.I. Szabo, Positive definite Berwald spaces, *Tensor N.S.*, **35** (1981), 25-39.