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CANONICAL ALMOST GEODESIC MAPPINGS OF THE FIRST TYPE OF SPACES WITH AFFINE CONNECTION ONTO GENERALIZED 2-RICCI-SYMMETRIC SPACES

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Abstract. In the paper we consider almost geodesic mappings of the first type of spaces with affine connections onto generalized 2-Ricci-symmetric spaces. The main equations for the mappings are obtained as a closed system of linear differential equations of Cauchy type in the covariant derivatives. The obtained result extends an amount of research produced by N.S. Sinyukov, V.E. Berezovski, J. Mikeš.

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1. Introduction

In the sixties of the 20th century has N.S. Sinyukov considered almost geodesic mappings of Riemannian and affinely connected spaces. The main results were presented in a research [12].

Theory goes back in 19th century by T. Levi-Civita, in which the problem on the search for Riemannian spaces with common geodesics was stated and solved in a special coordinate system. We note a remarkable fact that this problem is related to the study of equations of dynamics of mechanical systems.

N.S. Sinyukov specified three types of almost geodesic mappings π_1, π_2, π_3. The problem of completeness of classification had long remained unresolved. V.E. Berezovski and J. Mikeš [2] proved that for $n > 5$ other types of almost geodesic mappings except π_1, π_2, and π_3 do not exist. The authors have found conditions for the almost geodesic mappings π_1, π_2, π_3 intersect. It is proved that if an almost geodesic mapping f is simultaneously π_1 and π_2, then f is a mapping of affinely connected spaces with preserved linear complex of geodesic lines. If the mapping f is simultaneously π_1 and π_3, then f is a mapping of affinely connected spaces with preserved quadratic complex of geodesic lines.

The theory of almost geodesic mappings was developed by V.S. Sobchuk [13], N.Y. Yablonskaya [15], V.E. Berezovski, J. Mikeš [2–7, 9], O. Belova, J. Mikeš [1], N. Vesić, Lj.S. Velimirović, M.S. Stankovič [11, 14], A. Kozak, A. Borowiec [8] et al.

That direction is developing according to the Petrov's plan to build models of physical processes using mappings and transformations, see [10, 12].

N.S. Sinyukov [12] proved that the main equations for canonical almost geodesic mappings of spaces with affine connections onto Ricci-symmetric spaces can be written as a closed system of partial differential equations of Cauchy type in covariant derivatives. It follows that the general solution of the system depends on a finite number of essential parameters. J. Mikeš [9, 10] studied geodesic and holomorphically projective mappings of similar spaces.

The results were extended by V.E. Berezovski and J. Mikeš [6] to the cases of canonical almost geodesic mappings of the first type of spaces with affine connections onto Riemannian spaces and canonical almost geodesic mappings of the first type of spaces with affine connections onto generalized Ricci-symmetric spaces.

The paper is devoted to a study of canonical almost geodesic mappings of type π_1 of spaces with affine connections onto generalized 2-Ricci-symmetric spaces. The main equations for the mappings are obtained as a closed system of PDEs of Cauchy type. We have found the maximum number of essential parameters which the general solution of the system depends on.

2. Basic definitions of almost geodesic mappings of spaces with affine connections.

Let us recall the basic definition, formulas and theorems of the theory presented in [10, 12].

Consider an n-dimensional space A_n with affine connection Γ_{ij}^h without torsion. The space is referred to coordinates $x^1, x^2, \dots, x^n$. We assume that all functions under consideration are sufficiently differentiable, and we suppose that $n > 2$.

A curve defined in a space with affine connection is called *almost geodesic* if there exists a two-dimensional (differentiable) distribution D parallel along the curve (relative to the affine connection) such that for any tangent vector of the curve its parallel translation along the curve belongs to the distribution D [10].

A mapping $f\colon A_n \to \overline{A}_n$ is called *almost geodesic* if any geodesic curve of A_n is mapped under f onto an almost geodesic curve in $\overline{A}_n$.

Suppose, that a space A_n with affine connection $\Gamma^h_{ij}(x)$ admits a mapping f onto a space $\overline{A}_n$ with affine connection $\overline{\Gamma}^h_{ij}(x)$, and the spaces are referred to a common coordinate system $x^1, x^2, \dots, x^n$ with respect to the mapping.

The tensor

$$P^h_{ij}(x) = \overline{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x) \qquad (1)$$

is called a *deformation tensor* of the connections $\Gamma^h_{ij}(x)$ and $\overline{\Gamma}^h_{ij}(x)$ with respect to the mapping f. The symbols $\Gamma^h_{ij}(x)$ and $\overline{\Gamma}^h_{ij}(x)$ are components of affine connections of the spaces A_n and $\overline{A}_n$ respectively.

It is known [12] that in order that a mapping of a space A_n onto a space $\overline{A}_n$ be almost geodesic, it is necessary and sufficient that the deformation tensor $P^h_{ij}(x)$ of the mapping f in the common coordinate system $x^1, x^2, \dots, x^n$ has to satisfy the condition

$$A^h_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma = a \cdot P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + b \cdot \lambda^h,$$

where λ^h is an arbitrary vector, a and b are certain functions of variables $x^1, x^2, \dots, x^n$ and $\lambda^1, \lambda^2, \dots, \lambda^n$. The tensor A^h_{ijk} is defined as

$$A^h_{ijk} \stackrel{\text{def}}{=} P^h_{ij,k} + P^\alpha_{ij}P^h_{\alpha k}.$$

where the comma " , " denotes the covariant derivative with respect to the connection of the space A_n.

N.S. Sinyukov [12] distinguished three kinds of almost geodesic mappings, namely π_1, π_2 and π_3, characterized by following conditions for the deformation tensor.

A mapping $f : A_n \to \overline{A}_n$ is called an almost geodesic of type π_1, if the conditions

$$A^h_{(ijk)} = \delta^h_{(i}a_{jk)} + b_{(i}P^h_{jk)}, \qquad (2)$$

are satisfied. We denote by the parentheses an operation called *symmetrization* without division with respect to the indices i, j and k. Also a_{ij} are a certain symmetric tensor and a certain covariant vector respectively, and δ^h_i is the Kronecker delta.

If in the equation (2) the condition $b_i \equiv 0$ holds, then the almost geodesic mapping π_1 is called *canonical*.

It is known [12] that any almost geodesic mapping π_1 may be written as the composition of a canonical almost geodesic mapping of type π_1 and a geodesic mapping.

A mapping $f\colon A_n \to \overline{A}_n$ is called an almost geodesic of type π_2, if the conditions

$$P^h_{ij} = \delta^h_{(i}\psi_{j)} + F^h_{(i}\varphi_{j)},$$
$$F^h_{(i,j)} + F^h_\alpha F^\alpha_{(i}\varphi_{j)} = \delta^h_{(i}\mu_{j)} + F^h_{(i}\rho_{j)}$$

holds. Here ψ_i, φ_i, μ_i, ρ_i are certain covectors, F^h_i is a tensor of type $(1,1)$.

A mapping $f\colon A_n \to \overline{A}_n$ is called an almost geodesic of type π_3, if the conditions

$$P^h_{ij} = \delta^h_{(i}\psi_{j)} + \theta^h a_{ij},$$
$$\theta^h_{,i} = \rho \cdot \delta^h_i + \theta^h a_i$$

hold. Here θ^h is a torse forming vector, ψ_i, a_i are some covectors, a_{ij} is a symmetric tensor and ρ is an invariant.

As we have noted above, the mentioned types of mappings π_1, π_2 and π_3 can intersect. The classification completeness for dimension $n > 5$ has been proved in [2].

3. Ricci-symmetric, generalized Ricci-symmetric, 2-Ricci-symmetric and generalized 2-Ricci-symmetric spaces.

A space $\overline{A}_n$ with an affine connection is called *Ricci-symmetric* if its Ricci tensor $\overline{R}_{ij}$ satisfies the condition

$$\overline{R}_{ij;k} = 0.$$

The symbol " ; " denotes a covariant derivative with respect to the connection of the space $\overline{A}_n$.

In [12] N.S. Sinyukov considered canonical almost geodesic mappings of the first type of spaces with affine connections onto Riemannian Ricci-symmetric spaces. Taking into account the relations between the Riemannian tensors $\overline{R}^h_{ijk}$ and R^h_{ijk} of the spaces $\overline{A}_n$ and A_n respectively as follows from (1), the equations (2) for canonical almost geodesic mappings of the first type of spaces with affine connections were written in the form

$$3\big(P^h_{ij,k} + P^h_{k\alpha}P^\alpha_{ij}\big) = R^h_{(ij)k} - \overline{R}^h_{(ij)k} + \delta^h_{(k}a_{ij)}. \tag{3}$$

Considering (3) as a system of Cauchy type in A_n with respect to the functions P^h_{ij}, one has found the conditions of integrability of the system in the form

$$\overline{R}^h_{(ij)[k,l]} = R^h_{(ij)[k,l]} + \delta^h_{(i}a_{jk),l} - \delta^h_{(i}a_{jl),k} + 3\big(P^\alpha_{ij}\overline{R}^h_{\alpha kl} - P^h_{\alpha(j}R^\alpha_{i)kl}\big)$$
$$-P^h_{\alpha k}\big(R^h_{(ij)l} - \overline{R}^h_{(ij)l} + \delta^\alpha_{(i}a_{jl)}\big) + P^h_{\alpha l}\big(R^h_{(ij)k} - \overline{R}^h_{(ij)k} + \delta^\alpha_{(i}a_{jk)}\big),$$

where the brackets [ ] denote an operation called *antisymmetrization* (or, *alternation*) without division with respect to the mentioned indices.

Let us express in the left-hand side the covariant derivative with respect to the connection of A_n in terms of the covariant derivative with respect to the connection of $\overline{A}_n$. Taking account of (1), we obtain

$$\overline{R}^h_{(ij)[k;l]} = \delta^h_{(i}a_{jk),l} - \delta^h_{(i}a_{jl),k} + \theta^h_{ijkl}, \tag{4}$$

where

$$\theta^h_{ijkl} = R^h_{(ij)[k,l]} + 3\big(P^\alpha_{ij}\overline{R}^h_{\alpha kl} - P^h_{\alpha(j}R^\alpha_{i)kl}\big) - P^h_{\alpha k}\big(R^h_{(ij)l} - \overline{R}^h_{(ij)l} + \delta^\alpha_{(i}a_{jl)}\big)$$
$$+P^h_{\alpha l}\big(R^h_{(ij)k} - \overline{R}^h_{(ij)k} + \delta^\alpha_{(i}a_{jk)}\big) - P^\alpha_{l(i}\overline{R}^h_{|\alpha|j)k} - P^\alpha_{l(i}\overline{R}^h_{j)\alpha k} + P^\alpha_{k(i}\overline{R}^h_{|\alpha|j)l} + P^\alpha_{k(i}\overline{R}^h_{j)\alpha l}.$$

Here the symbol $|\alpha|$ means that in applying symmetrization the index α is omitted. Using the Ricci identity, let us write the conditions (4) in the form

$$\overline{R}^h_{ilk;j} + \overline{R}^h_{jlk;i} = \delta^h_{(i}a_{jk),l} - \delta^h_{(i}a_{jl),k} + \theta^h_{ijkl}. \tag{5}$$

Contracting (5) for h and k, we get the relations for covariant derivatives of the Ricci tensor $\overline{R}_{ij}$ for the space $\overline{A}_n$

$$\overline{R}_{il;j} + \overline{R}_{jl;i} = (n+1)a_{ij,l} - a_{l(i,j)} + \theta^\alpha_{ij\alpha l}. \tag{6}$$

A space $\overline{A}_n$ with an affine connection is called *generalized Ricci-symmetric* if the Ricci tensor $\overline{R}_{ij}$ for the space satisfies the condition

$$\overline{R}_{ij;k} + \overline{R}_{kj;i} = 0.$$

The concept of generalized Ricci-symmetric spaces was first introduced in [6]. The papers are devoted to the study of canonical almost geodesic mappings of spaces with affine connections onto the above mentioned spaces. It is proved that the main equations for the mappings can be obtained as a closed system of Cauchy-type differential equations in covariant derivatives.

A space $\overline{A}_n$ with an affine connection is called *2-Ricci-symmetric* if the Ricci tensor $\overline{R}_{ij}$ for the space satisfies the condition

$$\overline{R}_{ij;km} = 0.$$

A space $\overline{A}_n$ with an affine connection is called *generalized 2-Ricci-symmetric* if its Ricci tensor $\overline{R}_{ij}$ satisfies the condition

$$\overline{R}_{ij;km} + \overline{R}_{kj;im} = 0. \tag{7}$$

4. Canonical almost geodesic mappings of type π_1 of spaces with affine connections onto generalized 2-Ricci-symmetric spaces

The equation (6) was obtained for a canonical almost geodesic mapping of type π_1 of a space with an affine connection A_n onto another space with an affine connection $\overline{A}_n$.

First we differentiating covariantly the equation (6) with respect to the connection of the space A_n. Then taking account of (1), let us applicate the formulas

$$\begin{aligned}\left(\overline{R}_{il;j}\right)_{,k} &= \overline{R}_{il;jk} + \overline{R}_{\alpha l;j}P^{\alpha}_{ik} + \overline{R}_{i\alpha;j}P^{\alpha}_{lk} + \overline{R}_{il;\alpha}P^{\alpha}_{jk},\\ \overline{R}_{ij;k} &= \overline{R}_{ij,k} - \overline{R}_{i\alpha}P^{\alpha}_{jk} - \overline{R}_{\alpha j}P^{\alpha}_{ik}.\end{aligned}$$

Finally, we get

$$\overline{R}_{il;jk} + \overline{R}_{jl;ik} = (n+1)a_{ij,lk} - a_{li,jk} - a_{lj,ik} + C_{ijlk}, \tag{8}$$

where

$$\begin{aligned}C_{ijlk} = -\theta^{\alpha}_{ij\alpha l,k} &- \big(\overline{R}_{\alpha l,j} + \overline{R}_{jl,\alpha} - 2\overline{R}_{\beta l}P^{\beta}_{\alpha j} - \overline{R}_{\alpha\beta}P^{\beta}_{lj} - \overline{R}_{j\beta}P^{\beta}_{l\alpha}\big)P^{\alpha}_{ik}\\ &-\big(\overline{R}_{i\alpha,j} + \overline{R}_{j\alpha,i} - 2\overline{R}_{\beta\alpha}P^{\beta}_{ij} - \overline{R}_{i\beta}P^{\beta}_{\alpha j} - \overline{R}_{j\beta}P^{\beta}_{\alpha i}\big)P^{\alpha}_{lk}\\ &-\big(\overline{R}_{il,\alpha} + \overline{R}_{\alpha l,i} - 2\overline{R}_{\beta l}P^{\beta}_{\alpha i} - \overline{R}_{i\beta}P^{\beta}_{l\alpha} - \overline{R}_{\alpha\beta}P^{\beta}_{li}\big)P^{\alpha}_{jk}.\end{aligned} \tag{9}$$

Moreover, let us consider canonical almost geodesic mappings of type π_1 of spaces A_n with affine connections onto generalized 2-Ricci-symmetric spaces $\overline{A}_n$. Hence the Ricci tensor $\overline{R}_{ij}$ for the space $\overline{A}_n$ satisfied the conditions. Then the equation (8) may be written in the form

$$(n+1)a_{ij,lk} - a_{li,jk} - a_{lj,ik} = -C_{ijlk}, \tag{10}$$

when C_{ijlk} is defined by the formulas (9).

Let us interchange the indices l and j in (10) and sum for i and j. Then we have

$$a_{li,jk} + a_{lj,ik} = -\frac{1}{n}C_{(i|l|j)k} + \frac{2}{n}a_{ij,lk}. \tag{11}$$

The equation (10) by means of (11) can be written in the form

$$\frac{n^2+n-2}{n}a_{ij,lk} = -C_{ijlk} - \frac{1}{n}C_{(i|l|j)k}. \tag{12}$$

In the following we have assumed that a space A_n with an affine connection is given. Then taking account of the structure of the tensor C_{ijlk} which was determined by the formula (9), we see that the left hand side of the equation (12) depends on unknown functions

Differentiate covariantly the conditions of integrability (4) with respect to the connection of the space $\overline{A}_n$. Then express in the right-hand side the covariant derivative with respect to the connection of $\overline{A}_n$ in terms of the covariant derivative with respect to the connection of A_n. When we make use of the Ricci identity, we obtain

$$\overline{R}^h_{(ij)k;lm} - \overline{R}^h_{(ij)l;mk} = \delta^h_{(i}a_{jk),lm} - \delta^h_{(i}a_{jl),km} + T^h_{ijklm}, \tag{13}$$

where

$$\begin{aligned} T^h_{ijklm} = {} & \overline{R}^h_{\alpha mk}\overline{R}^\alpha_{(ij)l} - \overline{R}^\alpha_{lmk}\overline{R}^h_{(ij)\alpha} - \overline{R}^\alpha_{jmk}\overline{R}^h_{(i\alpha)l} - \overline{R}^\alpha_{imk}\overline{R}^h_{(j\alpha)l} - P^h_{m\alpha}\delta^\alpha_{(i}a_{jk),l} \\ & - P^\alpha_{mj}\delta^h_{(i}a_{\alpha k),l} - P^\alpha_{mi}\delta^h_{(\alpha}a_{jk),l} - P^\alpha_{mk}\delta^h_{(\alpha}a_{ij),l} - P^\alpha_{ml}\delta^h_{(i}a_{jk),\alpha} - P^h_{m\alpha}\delta^\alpha_{(i}a_{jl),k} \\ & + P^\alpha_{mi}\delta^h_{(\alpha}a_{jl),k} + P^\alpha_{mj}\delta^h_{(i}a_{\alpha l),k} + P^\alpha_{mk}\delta^h_{(i}a_{jl),\alpha} - P^\alpha_{ml}\delta^h_{(i}a_{j\alpha),k} - \theta^h_{ijkl,m} \\ & + P^h_{\alpha m}\theta^\alpha_{ijkl} - P^\alpha_{mi}\theta^h_{\alpha jkl} - P^\alpha_{mj}\theta^h_{i\alpha kl} - P^\alpha_{mk}\theta^h_{ij\alpha l} - P^\alpha_{ml}\theta^h_{ijk\alpha}. \end{aligned}$$

Alternating the equations (13) with respect to the indicies l and m, we obtain

$$\begin{aligned} \overline{R}^h_{(ij)m;lk} - \overline{R}^h_{(ij)l;mk} = {} & \delta^h_{(i}a_{jm),kl} - \delta^h_{(i}a_{jl),km} + T^h_{ijk[lm]} \\ & + \overline{R}^h_{(i|\alpha k|}\overline{R}^\alpha_{j)ml} + \overline{R}^h_{(ij)\alpha}\overline{R}^\alpha_{kml} - \overline{R}^\alpha_{(ij)k}\overline{R}^h_{\alpha ml} + \overline{R}^h_{\alpha(i|k|}\overline{R}^\alpha_{j)ml} \\ & + \delta^h_{(\alpha}a_{jk)}R^\alpha_{ilm} + \delta^h_{(\alpha}a_{ik)}R^\alpha_{jlm} + \delta^h_{(i}a_{j\alpha)}R^\alpha_{klm} - \delta^\alpha_{(i}a_{jk)}R^h_{\alpha lm}. \end{aligned} \tag{14}$$

Taking account of the properties of a curvature tensor R^h_{ijk}, we may write the conditions (14) in the form

$$\overline{R}^h_{iml;jk} + \overline{R}^h_{jml;ik} = \delta^h_{(i}a_{jl),km} - \delta^h_{(i}a_{jm),kl} - N^h_{ijklm}, \tag{15}$$

where

$$\begin{aligned} N^h_{ijklm} = {} & T^h_{ijk[lm]} + \overline{R}^\alpha_{iml}\overline{R}^h_{(\alpha j)k} + \overline{R}^\alpha_{jml}\overline{R}^h_{(\alpha i)k} + \overline{R}^\alpha_{kml}\overline{R}^h_{(ij)\alpha} - \overline{R}^h_{\alpha ml}\overline{R}^\alpha_{(ij)k} \\ & + \delta^h_{(\alpha}a_{jk)}R^\alpha_{ilm} + \delta^h_{(\alpha}a_{ik)}R^\alpha_{jlm} + \delta^h_{(\alpha}a_{ji)}R^\alpha_{klm} - a_{(ij}R^h_{k)lm}. \end{aligned}$$

Alternating the equations (15) with respect to the indicies j and k, we get

$$\begin{aligned} \overline{R}^h_{jml;ik} - \overline{R}^h_{kml;ij} = {} & \delta^h_{(i}a_{jl),km} - \delta^h_{(i}a_{jm),kl} - \delta^h_{(i}a_{kl),jm} + \delta^h_{(i}a_{km),jl} \\ & - N^h_{i[jk]lm} + \overline{R}^h_{\alpha ml}\overline{R}^\alpha_{ikj} + \overline{R}^h_{i\alpha l}\overline{R}^\alpha_{mkj} + \overline{R}^h_{im\alpha}\overline{R}^\alpha_{lkj} - \overline{R}^\alpha_{iml}\overline{R}^h_{\alpha kj}. \end{aligned} \tag{16}$$

Let us interchange i and k in (15) and subtract it from (16). Then we have

$$\begin{aligned} 2\overline{R}^h_{jml;ik} = {} & \delta^h_{(i}a_{jl),km} - \delta^h_{(i}a_{jm),kl} + \delta^h_{(k}a_{jm),il} + \delta^h_{(i}a_{km),jl} \\ & - \delta^h_{(i}a_{kl),jm} - \delta^h_{(k}a_{jl),im} + \Omega^h_{ijklm}. \end{aligned} \tag{17}$$

where

$$\begin{aligned}\Omega^h_{ijklm} = N^h_{kjilm} - N^h_{i[jk]lm} + \overline{R}^h_{\alpha ml}\overline{R}^\alpha_{ikj} + \overline{R}^h_{i\alpha l}\overline{R}^\alpha_{mkj} + \overline{R}^h_{im\alpha}\overline{R}^\alpha_{lkj}\\ -\overline{R}^\alpha_{iml}\overline{R}^h_{\alpha kj} - \overline{R}^\alpha_{kml}\overline{R}^h_{\alpha ji} + \overline{R}^h_{\alpha ml}\overline{R}^\alpha_{kji} + \overline{R}^h_{k\alpha l}\overline{R}^\alpha_{mji} + \overline{R}^h_{km\alpha}\overline{R}^\alpha_{lji}\\ -\overline{R}^\alpha_{jml}\overline{R}^h_{\alpha ik} + \overline{R}^h_{\alpha ml}\overline{R}^\alpha_{jik} + \overline{R}^h_{j\alpha l}\overline{R}^\alpha_{mik} + \overline{R}^h_{jm\alpha}\overline{R}^\alpha_{lik}.\end{aligned}$$

Let us express in the left-hand side of the equation (17) the covariant derivative with respect to the connection of $\overline{A}_n$ in terms of the covariant derivative with respect to the connection of A_n. We have

$$\begin{aligned}2\overline{R}^h_{jml,ik} = \delta^h_{(i}a_{jl),km} - \delta^h_{(i}a_{jm),kl} + \delta^h_{(k}a_{jm),il} + \delta^h_{(i}a_{km),jl}\\ -\delta^h_{(i}a_{kl),jm} - \delta^h_{(k}a_{jl),im} + S^h_{ijklm},\end{aligned} \tag{18}$$

where $S^h_{ijklm} = \Omega^h_{ijklm} - 2\big[\overline{R}^\alpha_{jml,i}P^h_{\alpha k}$

$$\begin{aligned}-\overline{R}^h_{\alpha ml,i}P^\alpha_{jk} - \overline{R}^h_{j\alpha l,i}P^\alpha_{mk} - \overline{R}^h_{jm\alpha,i}P^\alpha_{lk} - \overline{R}^h_{jml,\alpha}P^\alpha_{ik}\\ +\big(\overline{R}^\alpha_{jml}P^\beta_{\alpha i} - \overline{R}^\beta_{\alpha ml}P^\alpha_{ij} - \overline{R}^\beta_{j\alpha l}P^\alpha_{im} - \overline{R}^\beta_{jm\alpha}P^\alpha_{il}\big)P^h_{\beta k}\\ -\big(\overline{R}^\alpha_{jml}P^h_{\alpha\beta} - \overline{R}^h_{\alpha ml}P^\alpha_{\beta j} - \overline{R}^h_{j\alpha l}P^\alpha_{\beta m} - \overline{R}^h_{jm\alpha}P^\alpha_{\beta l}\big)P^\beta_{ik}\\ -\big(\overline{R}^\alpha_{\beta ml}P^h_{\alpha i} - \overline{R}^h_{\alpha ml}P^\alpha_{\beta i} - \overline{R}^h_{\beta\alpha l}P^\alpha_{im} - \overline{R}^h_{\beta m\alpha}P^\alpha_{il}\big)P^\beta_{jk}\\ -\big(\overline{R}^\alpha_{j\beta l}P^h_{\alpha i} - \overline{R}^h_{\alpha\beta l}P^\alpha_{ji} - \overline{R}^h_{j\alpha l}P^\alpha_{\beta i} - \overline{R}^h_{j\beta\alpha}P^\alpha_{il}\big)P^\beta_{km}\\ -\big(\overline{R}^\alpha_{jm\beta}P^h_{\alpha i} - \overline{R}^h_{\alpha m\beta}P^\alpha_{ji} - \overline{R}^h_{j\alpha\beta}P^\alpha_{mi} - \overline{R}^h_{jm\alpha}P^\alpha_{\beta i}\big)P^\beta_{kl}\big].\end{aligned}$$

We introduce the tensors a_{ijk} and $\overline{R}^h_{ijkm}$ defined by

$$a_{ij,k} = a_{ijk}, \tag{19}$$

$$\overline{R}^h_{ijk,m} = \overline{R}^h_{ijkm}. \tag{20}$$

Taking account of (20) we may write the equation (18) in the form

$$\begin{aligned}2\overline{R}^h_{jmli,k} = \delta^h_{(i}a_{jl),km} - \delta^h_{(i}a_{jm),kl} + \delta^h_{(k}a_{jm),il} + \delta^h_{(i}a_{km),jl}\\ -\delta^h_{(i}a_{kl),jm} - \delta^h_{(k}a_{jl),im} + S^h_{ijklm}.\end{aligned} \tag{21}$$

In the following we have assumed that in the left-hand side of the equation (21) the second order covariant derivatives are expressed according to (12). Taking acoount of (19), the equation (12) may be put in the form

$$\frac{n^2+n-2}{n}\,a_{ijl,k} = -C_{ijlk} - \frac{1}{n}\,C_{(i|l|j)k}. \tag{22}$$

We have assumed that in the right-hand sides of the equations (21) and (22) the covariant derivatives of tensors a_{ij} and $\overline{R}^h_{ijk}$ with respect to the connection of the space A_n are expressed according to (19) and (20).

Obviously, in the space A_n the equations (3), (19), (20), (21) and (22) form a closed system of PDEs of Cauchy type with respect to the functions $P^h_{ij}(x)$, $a_{ij}(x)$, $\overline{R}^h_{ijk}(x)$, $a_{ijk}(x)$, $\overline{R}^h_{ijkm}(x)$. The functions must satisfy the following algebraic conditions

$$\begin{aligned} P^h_{ij}(x) = P^h_{ji}(x), \qquad & a_{ij}(x) = a_{ji}(x), \\ \overline{R}^h_{i(jk)}(x) = \overline{R}^h_{(ijk)}(x) = 0, \qquad & \overline{R}^h_{i(jk)l}(x) = \overline{R}^h_{(ijk)l}(x) = 0. \end{aligned} \tag{23}$$

Hence we have proved

Theorem. *In order that a space A_n with an affine connection admit a canonical almost geodesic mapping of type π_1 onto a generalized 2-Ricci-symmetric space $\overline{A}_n$, it is necessary and sufficient that the mixed system of differential equations of Cauchy type in covariant derivatives* (3), (19), (20), (21), (22) *and* (23) *have a solution with respect to the unknown functions $P^h_{ij}(x)$, $a_{ij}(x)$, $\overline{R}^h_{ijk}(x)$, $a_{ijk}(x)$ and $\overline{R}^h_{ijkm}(x)$.*

And finely we obtain the corollary

Corollary. *The family of all generalized 2-Ricci-symmetric spaces which are images of a given space A_n with an affine connection respect to canonical almost geodesic mappings of type π_1 depend on no more then $\frac{1}{2}\, n\,(n+1)\,(n^2+n+1)$ essential parameters.*

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