Conharmonic Transformations of Locally Conformal Kähler Manifolds

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A Hermitian manifold (M^{2m}, J, g) is called a *locally conformal Kähler manifold (LCK - manifold)* if there is an open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ of M^{2m} and a family $\{\sigma_\alpha\}_{\alpha \in A}$ of C^∞ functions $\sigma_\alpha : U_\alpha \to \mathbb{R}$ so that each local metric

$$\hat{g}_\alpha = e^{-2\sigma_\alpha} g|_{U_\alpha}$$

is Kählerian. An LCK - manifold is endowed with some form ω, so called *Lee form* which can be calculated as [1]

$$\omega = \frac{1}{m-1} \delta\Omega \circ J.$$

The form should be closed:

$$d\omega = 0.$$

Here and below, we denote by comma covariant differentiation with respect to the Levi-Civita connection of (M^{2m}, J, g).

If a contravariant analitic vector field ξ generates conformal infinitesimal transformation of an LCK-manifold, then the field satisfy the system [2]

$$\begin{array}{ll} 1) & \xi_{i,j} = \xi_{ij}; \\ 2) & \xi_{i,j} + \xi_{j,i} = \big(\omega_\alpha \xi^\alpha + C\big) g_{ij}; \\ 3) & \xi_{i,jk} = \xi_\alpha R^\alpha_{kji} + \frac{1}{2}\Big(\big(\omega_\alpha \xi^\alpha\big)_{,k} g_{ij} + \big(\omega_\alpha \xi^\alpha\big)_{,j} g_{ik} - \big(\omega_\alpha \xi^\alpha\big)_{,i} g_{jk}\Big); \\ 4) & J^i_{j,k} \xi^k - J^\alpha_j \xi^i_{,\alpha} + J^i_\alpha \xi^\alpha_{,j} = 0. \end{array} \tag{1}$$

If a conformal transformation (1) also preserves a product Rg_{ij}, i. e. the equation

$$\mathfrak{L}_\xi \big(R g_{ij}\big) = 0 \tag{2}$$

holds, then the transformation is called *conharmonic.* We obtain the theorem.

Theorem 1. *If an LCK-manifold (M^{2m}, J, g) of non-zero scalar curvature admits nontrivial conharmonic transformations, then the general solution of the PDE system* (1)-(2) *depends on no more than $m^2 + 2m$ essential parameters.*

Also we have proved that the tensor

$$P_{ij} \overset{\text{def}}{=} \frac{1}{n-2}R_{ij} - \frac{1}{2}\omega_{i,j} - \frac{1}{4}\omega_i\omega_j + \frac{1}{8}\omega^{\alpha}\omega_{\alpha}g_{ij}$$

is preserved by conharmonic transformations.

References

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