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INFINITESIMAL CONFORMAL TRANSFORMATIONS OF RIEMANNIAN MANIFOLDS WHICH PRESERVE THE GENERALIZED EINSTEIN TENSOR

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Abstract. We study conformal transformations leaving the generalized Einstein tensor invariant. We have obtained the system of partial differential equations for the transformations, and explored its integrability conditions. Hence we have got the necessary and sufficient conditions in order that a manifold admit a group of conformal motions, preserving the generalized Einstein tensor. Also we have estimated the number of parameters which the group depends on.

Keywords: affine connection, Riemannian manifolds, generalized Einstein tensor, conformal transformation, Lie derivative

Mathematics subject classification: Primary 53B20; Secondary 53C21, 35G05

1 Introduction

Diffeomorphisms and transformations leaving certain geometric objects invariant are being given much attention of many researchers in the differential geometry realm. Finite conformal diffeomorfisms preserving the Einstein tensor

$$E_{ij} = R_{ij} - \frac{Rg_{ij}}{n}$$

were studied in [1]. Exploration of finite conformal diffeomorfisms preserving the stress-energy tensor

$$S_{ij} = R_{ij} - \frac{Rg_{ij}}{2}$$

by conformal mappings was presented in [6], [4]. Note that in many classical issues e. g. [9, p. 359], just the latter is referred to as the Einstein tensor. The tensor

$$\mathfrak{E}_{ij} \stackrel{\text{def}}{=} R_{ij} - \kappa R g_{ij}. \tag{1}$$

is referred to as **the generalized Einstein tensor**. Here κ is a constant. The conformal mappings preserving the the generalized Einstein tensor were explored in [3]. That paper is devoted to the study of infinitesimal conformal transformations which leave the generalized Einstein tensor invariant in the sense defined in [10].

2 Infinitesimal transformations of manifolds

Let (M^n, g) be a manifold of class C^∞ endowed with the nondegenerate quadratic differential form g called (pseudo-)Riemannian metric

$$g = g_{ij} dx^i dx^j,$$

covered by any system of coordinate neighbourhoods (x^i), where the indices $h, i, j \ldots$ run over the range $1, 2, \ldots, n$. The Riemann tensor, the Ricci tensor, and the scalar curvature are defined by the metric tensor g as follows:

$$R^h_{ijk} = \frac{\partial \Gamma^h_{ik}}{\partial x^j} - \frac{\partial \Gamma^h_{ij}}{\partial x^k} + \Gamma^\alpha_{ik}\Gamma^h_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^h_{\alpha k}, \qquad R_{ij} = R^\alpha_{ij\alpha}, \qquad R = R_{\alpha\beta} g^{\alpha\beta}.$$

Definition. *Transformation of a manifold* M^n

$$\overline{x}^h = x^h + \epsilon \xi^h(x^1, x^2, \ldots, x^n), \tag{2}$$

is called infinitesimal transformation *of a manifold* M^n*. Vector* $\xi(x^1, x^2, \ldots, x^n)$ *is often referred to as* a generator of transformation. *An arbitrary small parameter* ϵ *is independent on* x^i*.*

Lie derivative of a tensor of type (p, q) $T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ with respect to a vector field ξ may be calculated by using the formula [10, p. 14]:

$$\begin{aligned} \mathfrak{L}_\xi T^{i_1 \ldots i_p}_{j_1 \ldots j_q} = T^{i_1 \ldots i_p}_{j_1 \ldots j_q, s}\xi^s + T^{i_1 \ldots i_p}_{k j_2 \ldots j_q}\xi^k{}_{,j_1} + \ldots + T^{i_1 \ldots i_p}_{j_1 \ldots k}\xi^k{}_{,j_q} \\ - T^{l i_2 \ldots i_p}_{j_1 \ldots j_q}\xi^{i_1}{}_{,l} - \ldots - T^{i_1 i_2 \ldots l}_{j_1 \ldots j_q}\xi^{i_p}{}_{,l}. \end{aligned} \tag{3}$$

Hereinafter, the symbol "," denotes the covariant derivative with respect to the metric tensor of a manifold (M^n, g). Sometimes we also will denote the covariant derivative by the symbol "∇". In particular, for a metric tensor g_{ij} we get

$$\mathfrak{L}_\xi g_{ij} = \xi_{i,j} + \xi_{j,i} \tag{4}$$

If a manifold M^n is transformed then the metric tensor $\overline{g}$ of the transformed $\overline{M}^n$ is

$$\overline{g}_{ij} = g_{ij} + h_{ij}\epsilon, \tag{5}$$

where $h_{ij} = \mathfrak{L}_\xi g_{ij} = \xi_{i,j} + \xi_{j,i}$, and ϵ is the arbitrary small parameter mentioned in the **Definition** [5, p. 275]. For the Christoffel symbols we have also[10, p. 8]:

$$\mathfrak{L}_\xi \Gamma^h_{jk} = \nabla_k \nabla_j \xi^h + \xi^m R^h_{jmk} \tag{6}$$

Contracting (6) with g_{hi} we get:

$$\xi_{i,jk} = \xi_\alpha R^\alpha_{kji} + g_{hi}\mathfrak{L}_\xi \Gamma^h_{jk} \tag{7}$$

The item $g_{hi}\mathfrak{L}_\xi \Gamma^h_{jk}$ depends on transformation type.

3 Conformal infinitesimal transformations of Riemannian manifolds

Infinitesimal transformations are called conformal if the equations hold [5, p. 231]:

$$\mathfrak{L}_\xi g_{ij} = \xi_{i,j} + \xi_{j,i} = \varphi g_{ij}, \tag{8}$$

where φ is some function of the variables $x^1, x^2, \ldots, x^n$.

It is well known that, if a vector field ξ generates conformal infinitesimal transformations, then the field and the invariant φ satisfy the system [9], [7], [8, p. 253]:

$$\begin{array}{ll} 1) & \xi_{i,j} = \xi_{ij}; \\ 2) & \varphi_{,i} = \varphi_i; \\ 3) & \xi_{i,j} + \xi_{j,i} = \varphi g_{ij}; \\ 4) & \xi_{i,jk} = \xi_\alpha R^\alpha_{kji} + \frac{1}{2}(\varphi_k g_{ij} + \varphi_j g_{ik} - \varphi_i g_{jk}); \\ 5) & \varphi_{i,j} = \frac{2}{n-2}\left(\xi^\alpha R_{ij,\alpha} + \xi_{\alpha,i} R^\alpha_j + \xi_{\alpha,j} R^\alpha_i - \frac{g_{ij}}{2(n-1)}\left(\xi^\alpha R_{,\alpha} + \varphi R\right)\right). \end{array} \tag{9}$$

Taking account of (7) and (9_4), we get:

$$\mathfrak{L}_\xi \Gamma^h_{jk} = \frac{1}{2}(\varphi_j \delta^h_k + \varphi_k \delta^h_j - \varphi^h g_{jk}). \tag{10}$$

4 Conformal infinitesimal transformations of Riemannian manifolds, preserving the generalized Einstein tensor

If a manifold admits an infinitesimal transformation (2) preserving the generalized Einstein tensor then the Lie derivative of the tensor (1) vanishes:

$$\mathfrak{L}_\xi\big(R_{ij} - \kappa R g_{ij}\big) = 0. \tag{11}$$

Let us find the conditions of integrability of (9_4). According to [10, p. 17] for the Levi-Civita connection the conditions are

$$\mathfrak{L}_\xi R^h_{ijk} = \nabla_j \mathfrak{L}_\xi \Gamma^h_{ik} - \nabla_k \mathfrak{L}_\xi \Gamma^h_{ij}. \tag{12}$$

Substituting (10) into (12) we obtain

$$\begin{aligned} \mathfrak{L}_\xi R^h_{ijk} &= \frac{1}{2}\nabla_j(\varphi_i \delta^h_k + \varphi_k \delta^h_i - \varphi^h g_{ik}) - \frac{1}{2}\nabla_k(\varphi_i \delta^h_j + \varphi_j \delta^h_i - \varphi^h g_{ij}) \\ &= \frac{1}{2}\big(\delta^h_k \varphi_{i,j} - \delta^h_j \varphi_{i,k} + \nabla_k \varphi^h g_{ij} - \nabla_j \varphi^h g_{ik}\big). \end{aligned} \tag{13}$$

Let us contract (13) for h and k.

$$\mathfrak{L}_\xi R_{ij} = \frac{1}{2}\big((n-2)\varphi_{i,j} + \nabla_\alpha \varphi^\alpha g_{ij}\big). \tag{14}$$

On the other hand, taking account of (11), we can write (9_5) as

$$\varphi_{i,j} = \frac{2\mathfrak{L}_\xi\big(R g_{ij}\big)}{n-2}\left(\kappa - \frac{1}{2(n-1)}\right). \tag{15}$$

Since

$$\mathfrak{L}_\xi\big(R g_{ij}\big) = \big(\mathfrak{L}_\xi R + \varphi R\big) g_{ij}, \tag{16}$$

contracting (15) with g^{ij}, we have

$$\nabla_\alpha \varphi^\alpha = \frac{2n}{n-2} \cdot \frac{2n\kappa - 2\kappa - 1}{2(n-1)} \big(\mathfrak{L}_\xi R + \varphi R\big). \tag{17}$$

Substituting (15) and (17) in (14) and taking account of (16), we find

$$\begin{aligned} \mathfrak{L}_\xi R_{ij} = \frac{1}{2}\Big((n-2)\frac{2\mathfrak{L}_\xi\big(Rg_{ij}\big)}{n-2} \cdot \frac{2n\kappa - 2\kappa - 1}{2(n-1)} \\ + \frac{2n}{n-2} \cdot \frac{2n\kappa - 2\kappa - 1}{2(n-1)} \mathfrak{L}_\xi\big(Rg_{ij}\big)\Big). \end{aligned} \tag{18}$$

Simplifying in (18) the right hand side, we obtain

$$\mathfrak{L}_\xi R_{ij} = \frac{2n\kappa - 2\kappa - 1}{n-2} \mathfrak{L}_\xi\big(Rg_{ij}\big). \tag{19}$$

Comparing the equation (19) with (11) we see that it is possible to distinguish two cases:

a) $\kappa = \frac{1}{n}$;

b) $\kappa \neq \frac{1}{n}$;

The first case was considered in [2]. It is worthwhile noting that from the conditions $\kappa = \frac{1}{n}$ it follows that the tensor of concircular curvature should be invariant under the transformations:

$$\mathfrak{L}_\xi\Big(R^h_{jlk} - \frac{R}{n(n-1)}\big(\delta^h_k g_{jl} - \delta^h_l g_{jk}\big)\Big) = 0. \tag{20}$$

By the assumption that $\kappa \neq \frac{1}{n}$ the much stronger conditions are required:

$$\mathfrak{L}_\xi\big(R_{ij}\big) = 0; \tag{21}$$

and

$$\mathfrak{L}_\xi\big(Rg_{jk}\big) = 0. \tag{22}$$

It is obvious that from (21) follows (22). But in general the converse is not true. Taking account of (22), (15), we obtain that in consequence of (22), (15), the equations (9_5 can be written

$$\varphi_{i,j} = 0. \tag{23}$$

Substituting (23) in (13), we get

$$\mathfrak{L}_\xi R^h_{ijk} = 0. \tag{24}$$

Hence we obtain the theorem.

Theorem 4.1 *If a Riemannian manifold* (M^n, g), $(n > 3)$ *admits an infinitesimal conformal transformation which preserve the tensor* $\mathfrak{E}_{ij} = R_{ij} - \kappa R g_{ij}$ $\big(\kappa \neq \frac{1}{n}\big)$, *then the transformation also leaves invariant the Riemann tensor* R^h_{ijk}, *the Ricci tensor* R_{ij}, *and the product* Rg_{ij}.

For the case of conformal transformations the PDE system (9) becomes

$$\begin{array}{ll} 1) & \xi_{i,j} = \xi_{ij}; \\ 2) & \varphi_{,i} = \varphi_i; \\ 3) & \xi_{i,j} + \xi_{j,i} = \varphi g_{ij}; \\ 4) & \xi_{i,jk} = \xi_\alpha R^\alpha_{kji} + \frac{1}{2}(\varphi_k g_{ij} + \varphi_j g_{ik} - \varphi_i g_{jk}); \\ 5) & \varphi_{i,j} = 0. \end{array} \tag{25}$$

Since the manifold (M^n, g) admits a covariant constant vector field (23), according to a well-known theorem [10, p.75], we obtain the corollary.

Corollary *If a Riemannian manifold* (M^n, g), $(n > 3)$ *admits an infinitesimal conformal transformation which preserve the tensor* $\mathfrak{E}_{ij} = R_{ij} - \kappa R g_{ij}$ $\left(\kappa \neq \frac{1}{n}\right)$*, then there exists a coordinate system with respect to which the metric* g *of the manifold takes the form*

$$ds^2 = (dx^1)^2 + h_{ts} dx^t dx^s, \qquad t, s = \overline{2, n},$$

where coefficients h_{ts} *do not depend on* x^1.

Obviously, the conditions of integrability of (25_4) are (24). To find the conditions of integrability of (25_5) let us differentiate the equations covariantly with respect to x^k:

$$\nabla_k \nabla_j \varphi_i = 0. \tag{26}$$

Alternating (26) in j and k and using the Ricci identity, we obtain

$$\varphi_m R^m_{ijk} = 0. \tag{27}$$

The equations (26) are the conditions of integrability of (25_5).

Suppose that there are given two vectors $\underset{\alpha}{\xi^i}$ and $\underset{\beta}{\xi^i}$ satisfying the system (25). Let us refer to the corresponding scalars $\underset{\alpha}{\varphi}$ and $\underset{\beta}{\varphi}$, also satisfying the system (25), as the *associated scalars* to the vectors. Denote by $\underset{[\alpha,\beta]}{\xi} = [\underset{\alpha}{\xi}, \underset{\beta}{\xi}]$ their product (so called *commutator*), i. e.

$$\underset{[\alpha,\beta]}{\xi^h} = \underset{\alpha}{\xi^r} \partial_r \underset{\beta}{\xi^h} - \underset{\beta}{\xi^r} \partial_r \underset{\alpha}{\xi^h} = \underset{\alpha}{\xi^r} \nabla_r \underset{\beta}{\xi^h} - \underset{\beta}{\xi^r} \nabla_r \underset{\alpha}{\xi^h}.$$

Then, by virtue of the well-known formula (K. Yano, [11, pp. 20, 269])

$$\mathfrak{L}_{\underset{[\alpha,\beta]}{\xi}} \Gamma^h_{jk} = \mathfrak{L}_{\underset{\alpha}{\xi}} \mathfrak{L}_{\underset{\beta}{\xi}} \Gamma^h_{jk} - \mathfrak{L}_{\underset{\alpha}{\xi}} \mathfrak{L}_{\underset{\beta}{\xi}} \Gamma^h_{jk},$$

we find

$$\mathfrak{L}_{\underset{[\alpha,\beta]}{\xi}} \Gamma^h_{jk} = \frac{1}{2} \big(\underset{[\alpha,\beta]}{\varphi_j} \delta^h_k + \underset{[\alpha,\beta]}{\varphi_k} \delta^h_j - \underset{[\alpha,\beta]}{\varphi^h} g_{jk} \big), \tag{28}$$

where

$$\underset{[\alpha,\beta]}{\varphi} = \underset{\alpha}{\xi^r} \nabla_r \underset{\beta}{\varphi} - \underset{\beta}{\xi^r} \nabla_r \underset{\alpha}{\varphi} \tag{29}$$

and

$$\underset{[\alpha,\beta]}{\varphi_j} = \nabla_j \underset{[\alpha,\beta]}{\varphi} = \nabla_j \underset{\alpha}{\xi^r} \nabla_r \underset{\beta}{\varphi} - \nabla_j \underset{\beta}{\xi^r} \nabla_r \underset{\alpha}{\varphi}, \tag{30}$$

since the equation (25_5) holds (see also [9, p. 274]). Differentiating covariantly (30) with respect to x_k, then taking account of (26) and (25_4), we get

$$\begin{aligned} \nabla_k \underset{[\alpha,\beta]}{\varphi_j} &= \nabla_k \nabla_j \underset{[\alpha,\beta]}{\varphi} = \nabla_k \nabla_j \underset{\alpha}{\xi^r} \nabla_r \underset{\beta}{\varphi} - \nabla_k \nabla_j \underset{\beta}{\xi^r} \nabla_r \underset{\alpha}{\varphi} \\ &= \frac{1}{2} \big(\underset{\alpha}{\varphi_j} \delta^r_k + \underset{\alpha}{\varphi_k} \delta^r_j - \underset{\alpha}{\varphi^r} g_{jk} \big) \nabla_r \underset{\beta}{\varphi} - \frac{1}{2} \big(\underset{\beta}{\varphi_j} \delta^r_k + \underset{\beta}{\varphi_k} \delta^r_j - \underset{\beta}{\varphi^r} g_{jk} \big) \nabla_r \underset{\alpha}{\varphi} = 0. \end{aligned} \tag{31}$$

Also it is known that the vector $\underset{[\alpha,\beta]}{\xi^h}$ and its associated scalar $\underset{[\alpha,\beta]}{\varphi}$ satisfy the equations (25_3). Then from the equations (28), (31) it follows that the vector $\underset{[\alpha,\beta]}{\xi^h}$ generates a one-parameter continuous group of conformal transformations leaving the generalized Einstein tensor invariant. Thus we obtain the theorem.

Theorem 4.2 *On a Riemannian manifold* (M^n, g), $(n > 3)$ *which admits infinitesimal conformal transformations preserving the tensor* $\mathfrak{E}_{ij} = R_{ij} - \kappa R g_{ij}$ $(\kappa \neq \frac{1}{n})$, *the set of all vectors generating the transformations forms a Lie algebra.*

There are a very useful identity for a general tensor P^h_{ij} ([10, p. 16]):

$$\mathfrak{L}_\xi \nabla_k P^h_{ij} - \nabla_k \mathfrak{L}_\xi P^h_{ij} = P^t_{ij} \mathfrak{L}_\xi \Gamma^h_{tk} - P^h_{tj} \mathfrak{L}_\xi \Gamma^t_{ik} - P^h_{it} \mathfrak{L}_\xi \Gamma^t_{jk}. \tag{32}$$

Applying the formula (31) to the Riemann tensor, we have

$$\mathfrak{L}_\xi \nabla_l R^h_{ijk} - \nabla_l \mathfrak{L}_\xi R^h_{ijk} = R^t_{ijk} \mathfrak{L}_\xi \Gamma^h_{tl} - R^h_{tjk} \mathfrak{L}_\xi \Gamma^t_{il} - R^h_{itk} \mathfrak{L}_\xi \Gamma^t_{jl} - R^h_{ijt} \mathfrak{L}_\xi \Gamma^t_{kl}. \tag{33}$$

To find differential prolongations of (24), we substitute (24) and (10) in the identity (33)

$$\begin{aligned} \mathfrak{L}_\xi \nabla_l R^h_{ijk} = R^t_{ijk} \frac{1}{2}(\varphi_t \delta^h_l + \varphi_l \delta^h_t - \varphi^h g_{lt}) - R^h_{tjk} \frac{1}{2}(\varphi_i \delta^t_l + \varphi_l \delta^t_i - \varphi^t g_{il}) \\ - R^h_{itk} \frac{1}{2}(\varphi_j \delta^t_l + \varphi_l \delta^t_j - \varphi^t g_{jl}) - R^h_{ijt} \frac{1}{2}(\varphi_k \delta^t_l + \varphi_l \delta^t_k - \varphi^t g_{kl}). \end{aligned}$$

Simplifying and and taking account of (27), we obtain

$$\mathfrak{L}_\xi \nabla_l R^h_{ijk} = \varphi_l R^h_{ijk} + \frac{1}{2}\left(R^h_{ljk}\varphi_i + R^h_{ilk}\varphi_j + R^h_{ijl}\varphi_k + R_{lijk}\varphi^h\right). \tag{34}$$

The equations (34) are the first differential prolongations of (24). It is clear that in like manner, we can continue this process as for as we wish. To find differential prolongations of (27) differentiate the equations covariantly with respect to x^l_1. Taking account of (23), we obtain

$$\varphi_m R^m_{ijk,l_1} = 0. \tag{35}$$

Continuing this process, we get a sequence of differential prolongations

$$\begin{aligned} &\varphi_m R^m_{ijk,l_1 l_2} = 0, \\ &\dots \\ &\varphi_m R^m_{ijk,l_1 l_2 \dots l_r} = 0, \\ &\dots . \end{aligned} \tag{36}$$

We also can continue the process as for as we wish. Thus we get the theorem.

Theorem 4.3 *In order that a Riemannian manifold* (M^n, g) *admit a group of conformal transformation which preserve the tensor* $\mathfrak{E}_{ij} = R_{ij} - \kappa R g_{ij}$ $\left(\kappa \neq \frac{1}{n}\right)$, *is necessary and sufficient that the equations* (8), (24), (27), (34), (36), (36) ... *be algebraically consistent with respect to* φ, φ_i, ξ_i *and* $\xi_{i,j}$. *If there are, among the equations* (24), (27), (34), (36), (36) ..., *exactly* k *equations which arc linearly independent among themselves and of* (8), *then the manifold admits a* $\frac{(n+1)(n+2)}{2} - k$ *parameter group of conformal transformation which preserve the tensor* $\mathfrak{E}_{ij} = R_{ij} - \kappa R g_{ij}$ $\left(\kappa \neq \frac{1}{n}\right)$.

In order that a manifold (M^n, g) admit the group of the maximum order $\frac{(n+1)(n+2)}{2}$, it is necessary and sufficient that the equations (24) and (27) must be satisfied by any φ, φ_i, ξ_i and $\xi_{i,j}$ such that $\xi_{i,j} + \xi_{j,i} = \varphi g_{ij}$. From the arbitrariness of the φ_i we find

$$R^h_{ijk} = 0.$$

This means that only a manifold endowed with an Euclidean metric admits the group of the maximum order $\frac{(n+1)(n+2)}{2}$. Then the condition (11) is trivially satisfied.

5 Conformal infinitesimal transformations of Riemannian manifolds of non-zero scalar curvature which leave the generalized Einstein tensor invariant

Let us first consider the case where $R = const \neq 0$. In consequence of (16) we have

$$\mathfrak{L}_\xi\big(Rg_{ij}\big) = \big(\mathfrak{L}_\xi R + \varphi R\big)g_{ij} = \big(\xi^j\nabla_j R + \varphi R\big)g_{ij} = \varphi R g_{ij}.$$

Comparing this equation with (22) we have the theorem.

Theorem 5.1 *Riemannian manifolds of non-zero constant scalar curvature do not admit non-trivial conformal transformations preserving the generalized Einstein tensor* $\mathfrak{E}_{ij} = R_{ij} - \kappa R g_{ij}$, *(*$\kappa \neq \frac{1}{n}$*).*

We shall next consider the case in which $R \neq const$. In this case, in accordance with (16) and taking account of (22) we get

$$\xi^j\nabla_j R + \varphi R = 0.$$

Hence we have

$$\varphi = -\frac{\xi^j\nabla_j R}{R} = -\big(\xi^\alpha\nabla_\alpha(ln|R|)\big). \tag{37}$$

In this case, the PDE system (25) can be written as

$$\begin{array}{l} 1)\xi_{i,j} = \xi_{ij}; \\ 2)\xi_{i,j} + \xi_{j,i} = -\big(\xi^\alpha\nabla_\alpha(ln|R|)\big)g_{ij}; \\ 3)\xi_{i,jk} = \xi_\alpha R^\alpha_{kji} - \frac{1}{2}\big(\nabla_k\big(\xi^\alpha\nabla_\alpha(ln|R|)\big)g_{ij} \\ \quad +\nabla_j\big(\xi^\alpha\nabla_\alpha(ln|R|)\big)g_{ik} - \nabla_i\big(\xi^\alpha\nabla_\alpha(ln|R|)\big)g_{jk}\big). \end{array} \tag{38}$$

the integrability conditions of (25_3) are (24)

$$\mathfrak{L}_\xi R^h_{ijk} = \xi^t\nabla_t R^h_{ijk} - R^t_{ijk}\nabla_t\xi^h + R^h_{tjk}\nabla_i\xi^t + R^h_{itk}\nabla_j\xi^t + R^h_{ijt}\nabla_k\xi^t = 0. \tag{39}$$

their differential prolongations (34)

$$\mathfrak{L}_\xi\nabla_l R^h_{ijk} = \varphi_l R^h_{ijk} + \frac{1}{2}\big(R^h_{ljk}\varphi_i + R^h_{ilk}\varphi_j + R^h_{ijl}\varphi_k + R_{lijk}\varphi^h\big), \tag{40}$$

and so on. Here and throughout the remainder of the paper it is understood that

$$\varphi_i = -\nabla_i\big(\xi^\alpha\nabla_\alpha(ln|R|)\big) \quad \text{and} \quad \varphi^h = -g^{hi}\nabla_i\big(\xi^\alpha\nabla_\alpha(ln|R|).$$

Obviously, the general solution of the PDE system (38) depends on no more than $\frac{n(n+1)}{2}$ essential parameters. In order that a manifold admit a group of conformal transformations of the maximum order $\frac{n(n+1)}{2}$, it is necessary and sufficient that the equations (39) and (40) be identically satisfied by any ξ_i and $\xi_{i,j}$ such that $\xi_{i,j} + \xi_{j,i} = -\big(\xi^\alpha \nabla_\alpha (ln|R|)\big) g_{ij}$. From the arbitrariness of the ξ_i we find

$$\nabla_l R^h_{ijk} = 0.$$

Hence the equation (40) can be written as

$$\varphi_l R^h_{ijk} + \frac{1}{2}\big(R^h_{ljk}\varphi_i + R^h_{ilk}\varphi_j + R^h_{ijl}\varphi_k + R_{lijk}\varphi^h\big) = 0. \tag{41}$$

Contracting (40) with φ^l and and taking account of (27), we get:

$$||\varphi||^2 R^h_{ijk} = 0. \tag{42}$$

The equation contradicts our assumption that $R \neq 0$. Hence we get the theorem.

Theorem 5.2 *If a Riemannian manifold of non-zero scalar curvature admits nontrivial conformal transformations preserving the generalized Einstein tensor* $\mathfrak{E}_{ij} = R_{ij} - \kappa R g_{ij}$, *(*$\kappa \neq \frac{1}{n}$*), then the general solution of the PDE system* (38) *depends on less than* $\frac{n(n+1)}{2}$ *essential parameters.*

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