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Geodesic and almost geodesic mappings onto Ricci symmetric spaces

V. Berezovskii, P. Peška and J. Mikeš

Uman National University of Horticulture, Dept. of Mathematics
Instytutska 1, Uman, Ukraine
Department of Algebra and Geometry,
Faculty of Science, Palacký University Olomouc,
17. listopadu 1192/12, 774 16 Olomouc
Email: `berez.volod@rambler.ru`, `patrik_peska@seznam.cz`,
`josef.mikes@upol.cz`

Abstract: This paper is devoted to study of geodesic and almost geodesic mappings of special spaces with affine connection. In the first section, we mention the basic definition of geodesic and almost geodesic mappings. The next section is devoted to geodesic mappings onto Ricci symmetric manifolds and its fundamental diferential equation in Cauchy type form in covariant derivatives. We also study almost geodesic mappings of the first type onto symmetric space.

Keywords: geodesic mapping, almost geodesic mapping, spaces with affine connection, (pseudo-) Riemannian space

INTRODUCTION

This paper is dedicated to further development of theories of geodesic and almost geodesic mappings of spaces with affine connection on some special spaces, especially symmetric and Ricci symmetric spaces.

T. Levi-Civita [14] set and solved a special equation for the problem of finding Riemannian spaces with common geodesics. It is worth to note that it was connected with studying the equations of mechanical systems.

The theory of geodesic mappings has been developed later in the works by Thomas, Weyl, Shirokov, Solodovnikov, Sinyukov, Mikeš and others. Studying geodesic mappings was followed up in the works by Kagan, Vraceanu, Shapiro, Vedenyapin and others. The mentioned authors identified special classes $(n-1)$-projective spaces, see [15, 17, 18, 19, 26].

The quasi geodesic mappings introduced A.Z. Petrov [23]. Special quasi geodesic mappings, in particular, are holomorphically projective mappings of Kähler spaces, which first had been studied by Otsuki and Tashiro [22], Prvanovich [25] and others, see [16, 17, 18, 19, 26].

The natural generalizations of these classes are almost geodesic mappings, which were introduces by Sinyukov. He also defined three types of almost geodesic mappings π_1, π_2 and π_3, see [26]. The almost geodesic mappings has been developed later in the works by Sobchuk, Yablonskaya, Berezovskii, Mikeš, see [1, 2, 3, 4, 5, 6, 16, 18], [17], pp. 455–480.

Recently, some questions related to geodesic and almost geodesic mappings, and tehir generalizations have also been studied in [9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 21, 24, 27].

In this paper there were obtained the main equations of geodesic mappings of spaces with affine connection onto Ricci symmetric manifolds and almost geodesic mappings of the first type of spaces with affine connection onto symmetric spaces in a form of closed systems of Cauchy type in covariant derivatives. In this paper was also set the number of essential real parameters for the general solution of that system.

We assume that the studied spaces are simply connected with dimension $n > 2$, and we consider that geometric objects are continuous and smooth enough.

1 GEODESIC AND ALMOST GEODESIC MAPPINGS THEORY

Let us mention the following definition of geodesic and almost geodesic curves and mappings, see [17], pp. 88, 257, 455-458, [26], pp. 43, 70, 156-162.

1.1 Geodesics and geodesic mappings

It is known, the curve ℓ defined in space with affine connection A_n is called *geodesic* if there exists a parallel tangent vector field along it.

The diffeomorphism $f\colon A_n \to \bar{A}_n$ between spaces with affine connection is called *geodesic mapping* if any geodesic of A_n is mapped onto geodesic on $\bar{A}_n$.

The diffeomorphism f is geodesic mapping if and only if in a common coordinate system $x = (x^1, x^2, \dots, x^n)$ respective mapping f yields the Levi-Civita equation

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi_i \delta^h_j + \psi_j \delta^h_i, \tag{1}$$

where $\bar{\Gamma}^h_{ij}(x)$ and $\Gamma^h_{ij}(x)$ are components of affine connection of manifolds A_n and $\bar{A}_n$, respectively, ψ_i are components of covector, and δ^h_i is the Kronecker delta.

1.2 Almost geodesics curves and mappings

The curve ℓ defined in space with affine connection A_n is called *almost geodesic* if there exists two dimensional parallel plane along it, which contains its tangent vector.

The diffeomorphism $f\colon A_n \to \bar{A}_n$ is called *almost geodesic mapping (AGM)* if any geodesic of A_n is mapped onto almost geodesic in $\bar{A}_n$.

The diffeomorphism f is almost geodesic mapping if and only if in a common coordinate system $x = (x^1, x^2, \dots, x^n)$ respective mapping f the deformation tensor of connections

$$P^h_{ij}(x) = \bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x) \tag{2}$$

yields the Sinyukov's equation

$$A^h_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma = a\lambda^h + \lambda P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta,$$

where $A^h_{ijk} = P^h_{ij,k} + P^\alpha_{ij}P^h_{\alpha k}$, λ^h is a vector, a and b are functions of variables x^h and λ^h. Here and after a symbol " , " denotes a covariant derivative respective connection of A_n.

Sinyukov defined three types of almost geodesic mappings π_1, π_2 and π_3. We proved [1], that if dimension $n > 5$ no other types exist.

Almost geodesic mappings of type π_1 are characterized by the following conditions on deformation tensor

$$A^h_{(ijk)} = a_{(ij}\delta^h_{j)} + b_{(i}P^h_{jk)}, \tag{3}$$

where a_{ij} is a symmetric tensor, b_i is covector, and (i,j,k) is symmetrization of mentioned indices without division.

If in an equation (3) the covector b_i is vanishing, then the mapping is called *canonical*. It is known [26], p. 171, [17], p. 463, that any almost geodesic mappings of type π_1 can be regarded as a composition of canonical almost geodesic mapping of type π_1 and geodesic mapping.

2 THE GEODESIC MAPPINGS ONTO RICCI SYMMETRIC MANIFOLDS

In this section we will study the geodesic mappings of A_n onto Ricci symmetric spaces $\bar{A}_n$. The manifold with affine connection is called *Ricci symmetric* if the Ricci tensor in it is absolutely parallel, see [17], p. 87.

Therefore, Ricci symmetric manifold $\bar{A}_n$ is characterized by the condition

$$\bar{R}_{ij|k} \equiv 0, \tag{4}$$

where $\bar{R}_{ij}$ is the Ricci tensor of $\bar{A}_n$, the symbol " | " denotes a covariant derivative of connection on $\bar{A}_n$.

Because for Riemannian tensor (or curvature tensor) $\bar{R}$ of $\bar{A}_n$ holds

$$\bar{R}^h_{ijk|m} = \frac{\partial \bar{R}^h_{ijk}}{\partial x^m} + \bar{\Gamma}^h_{m\alpha}\bar{R}^\alpha_{ijk} - \bar{\Gamma}^\alpha_{mi}\bar{R}^h_{\alpha jk} - \bar{\Gamma}^\alpha_{mj}\bar{R}^h_{i\alpha k} - \bar{\Gamma}^\alpha_{mk}\bar{R}^h_{i\alpha j},$$

then from the formula (2) we obtain

$$\bar{R}^h_{ijk|m} = \bar{R}^h_{ijk,m} + P^h_{m\alpha}\bar{R}^\alpha_{ijk} - P^\alpha_{mi}\bar{R}^h_{\alpha jk} - P^\alpha_{mj}\bar{R}^h_{i\alpha k} - P^\alpha_{mk}\bar{R}^h_{ij\alpha}, \tag{5}$$

where $\bar{R}^h_{ijk}$ are components of Riemannian tensor $\bar{R}$.

Contracting formula (5) with respect to indices h and k, we obtain

$$\bar{R}_{ij|m} = \bar{R}_{ij,m} - P^\alpha_{mi}\bar{R}_{\alpha j} - P^\alpha_{mj}\bar{R}_{i\alpha}. \tag{6}$$

To follow, let us suppose the manifolds $\bar{A}_n$ is Ricci symmetric. Using the formula (4), we get

$$\bar{R}_{ij,m} = P^{\alpha}_{mi}\bar{R}_{\alpha j} + P^{\alpha}_{mj}\bar{R}_{i\alpha}. \tag{7}$$

Taking into consideration the Levi-Civita equation (1) and basing on the formula (7), we can write

$$\bar{R}_{ij,m} = 2\psi_m\bar{R}_{ij} + \psi_i\bar{R}_{mj} + \psi_j\bar{R}_{im}. \tag{8}$$

It is known, that between Riemannian tensors on A_n and $\bar{A}_n$ there is a dependence

$$\bar{R}^h_{ijk} = R^h_{ijk} + P^h_{ik,j} - P^h_{ij,k} + P^{\alpha}_{ik}P^h_{j\alpha} - P^{\alpha}_{ij}P^h_{k\alpha}, \tag{9}$$

where R^h_{ijk} are components of Riemannian tensor R on A_n.

Because deformation tensor P has the structure (1) from the formula (9) after computation we obtain

$$\bar{R}^h_{ijk} = R^h_{ijk} - \delta^h_j\psi_{i,k} + \delta^h_k\psi_{i,j} - \delta^h_i\psi_{j,k} + \delta^h_i\psi_{k,j} + \delta^h_j\psi_i\psi_k - \delta^h_k\psi_i\psi_j. \tag{10}$$

Let us contract (10) with respect to indices h and k. As the result we get

$$\bar{R}_{ij} = R_{ij} + n\psi_{i,j} - \psi_{j,i} + (1-n)\psi_i\psi_j. \tag{11}$$

From the equation (11), we obtain the following

$$\psi_{i,j} = \frac{1}{n^2-1}\left[n\bar{R}_{ij} + \bar{R}_{ji} - (nR_{ij} + R_{ji})\right] + \psi_i\psi_j. \tag{12}$$

The following theorem holds.

Theorem 1 *The manifold A_n admits geodesic mapping onto Ricci symmetric manifold $\bar{A}_n$ if and only if it consists a solution of closed system of Cauchy type equations in covariant derivative (8) and (12) with respect to unknown functions $\bar{R}_{ij}(x)$ and $\psi_i(x)$.*
General solution of the above system depends on at most than $n(n+1)$ essential real parameters.

Because the systems (8) and (12) have only one solution for the initial conditions in point x_0

$$\bar{R}_{ij}(x_0) \text{ and } \psi_i(x_0),$$

from this follows the above number of essential real parameter.

3 AGM OF THE FIRST TYPE ONTO SYMMETRIC SPACES

Now, let us consider canonical almost geodesic mappings of spaces with affine connection A_n onto symmetric space $\bar{A}_n$.

A space $\bar{A}_n$ with affine connection $\bar{\nabla}$ is called (locally) *symmetric* if Riemannian tensor in it is absolutely parallel (P.A. Shirokov, É. Cartan [7], S. Helgason [8], see [17], p. 286, [26], p. 42. Therefore, symmetric manifolds $\bar{A}_n$ are characterized by the condition

$$\bar{R}^h_{ijk|m} \equiv 0. \tag{13}$$

Further, let us suppose that manifold $\bar{A}_n$ is symmetric. Basing on the formula (13), from (5) we obtain

$$\bar{R}^h_{ijk,m} = P^\alpha_{mi}\bar{R}^h_{\alpha jk} + P^\alpha_{mj}\bar{R}^h_{i\alpha k} + P^\alpha_{mk}\bar{R}^h_{ij\alpha} - P^h_{m\alpha}\bar{R}^\alpha_{ijk}. \tag{14}$$

Formula (14) can be applied to general mappings of A_n onto symmetric spaces $\bar{A}_n$.

It is known that the equation (3) has the following form

$$3(P^h_{ij,k} + P^\alpha_{ij}P^h_{\alpha k}) = R^h_{(ij)k} - \bar{R}^h_{(ij)k} + P^\alpha_{mk}\bar{R}^h_{ij\alpha} + \delta^h_{(k}a_{ij)} + b_{(i}P^h_{jk)}. \tag{15}$$

From formula (15) of canonical almost geodesic mappings of the first type, we obtain the equation

$$P^h_{ij,k} = \frac{1}{3}\,(R^h_{(ij)k} - \bar{R}^h_{(ij)k} + \delta^h_{(k}a_{ij)}) - P^\alpha_{ij}P^h_{\alpha k}. \tag{16}$$

From the integrability condition of the equation (16) considering formulas (15) and (16), after computation, we get the following

$$\begin{aligned} \delta^h_{(m}a_{ij),k} - \delta^h_{(k}a_{ij),m} = -3P^h_{\alpha j}R^\alpha_{ikm} - 3P^h_{i\alpha}R^\alpha_{jkm} - R^\alpha_{(ij)m}P^h_{\alpha k} + \\ R^\alpha_{(ij)k}P^h_{\alpha m} + R^h_{(ij)k,m} - R^h_{(ij)m,k} + 3\bar{R}^h_{\alpha km}P^\alpha_{ij} - P^\alpha_{mi}\bar{R}^h_{(j\alpha)k} + \\ P^\alpha_{ki}\bar{R}^h_{(j\alpha)m} - P^\alpha_{mj}\bar{R}^h_{(i\alpha)k} + P^\alpha_{kj}\bar{R}^h_{(i\alpha)m} - \delta^\alpha_{(m}a_{ij)}P^h_{\alpha k} + \delta^\alpha_{(k}a_{ij)}P^h_{\alpha m}. \end{aligned} \tag{17}$$

After contracting the equation (17) with respect to indices h and m, we obtain

$$\begin{aligned} (n+1)a_{ij,k} - a_{ik,j} - a_{jk,i} = -3P^\beta_{\alpha(j}R^\alpha_{i)k\beta} - P^\beta_{\alpha k}R^\alpha_{(ij)\beta} + P^\beta_{\alpha\beta}R^\alpha_{(ij)k} + \\ R^\beta_{(ij)k,\beta} - R_{(ij),k} + 3P^\alpha_{ij}\bar{R}_{\alpha k} - P^\alpha_{\beta i}\bar{R}^\beta_{(j\alpha)k} + P^\alpha_{ki}\bar{R}_{j\alpha} - \\ P^\alpha_{\beta j}\bar{R}^\beta_{(i\alpha)k} + P^\alpha_{kj}\bar{R}_{(i\alpha)} - \delta^\alpha_{(\beta}a_{ij)}P^\beta_{\alpha k} + \delta^\alpha_{(k}a_{ij)}P^\beta_{\alpha\beta}. \end{aligned} \tag{18}$$

Let us alternate the equation (18) with respect to indices j and k. Then, we can write (18) in the following form

$$\begin{aligned} (n-1)a_{ij,k} = -3P^\beta_{\alpha(j}R^\alpha_{i)k\beta} - P^\beta_{\alpha k}R^\alpha_{(ij)\beta} + P^\beta_{\alpha\beta}R^\alpha_{(ij)k} + R^\beta_{(ij)k,\beta} - \\ R_{(ij),k} + 3P^\alpha_{ij}\bar{R}_{\alpha k} - P^\beta_{\alpha i}\bar{R}^\alpha_{(j\beta)k} + P^\alpha_{ki}\bar{R}_{\alpha j} - P^\beta_{\alpha j}\bar{R}^\alpha_{(i\beta)k} + \\ P^\alpha_{kj}\bar{R}_{(i\alpha)} - \delta^\alpha_{(\beta}a_{ij)}P^\beta_{\alpha k} + \delta^\alpha_{(k}a_{ij)}P^\beta_{\alpha\beta} - \tfrac{1}{n+2}\,B_{(ij)k}, \end{aligned} \tag{19}$$

where

$$\begin{aligned} B_{ijk} = P^\beta_{\alpha k}(R^\alpha_{ij\beta} + R^\alpha_{\beta ji}) - P^\beta_{\alpha k}(R^\alpha_{ik\beta} + R^\alpha_{\beta ki}) + 3P^\beta_{\alpha\beta}R^\alpha_{ijk} + 3R^\beta_{ijk,\beta} - \\ R_{(ij),k} + R_{(ik),j} + 2P^\alpha_{ij}\bar{R}_{\alpha k} - 2P^\alpha_{ik}\bar{R}_{\alpha j} + P^\alpha_{ki}\bar{R}_{j\alpha} - \\ P^\alpha_{ij}\bar{R}_{k\alpha} - P^\alpha_{\beta j}\bar{R}^\beta_{(i\alpha)k} + P^\alpha_{\beta k}\bar{R}^\beta_{(i\alpha)j} - a_{ij}P^\alpha_{\alpha k} - \\ a_{\alpha j}P^\alpha_{ik} + a_{ik}P^\alpha_{ik} + a_{\alpha k}P^\alpha_{ij}. \end{aligned} \tag{20}$$

It is evident, the equations (14), (16) and (19) in the given manifold A_n have a form of closed system of Cauchy type equation regarding unknown function $\bar{R}^h_{ijk}(x)$, $P^h_{ij}(x)$ and $a_{ij}(x)$ which also satisfies the algebraic conditions

$$\bar{R}^h_{i(jk)} = 0, \qquad \bar{R}^h_{(ijk)} = 0, \qquad P^h_{ij} = P^h_{ji}, \qquad a_{ij} = a_{ji}. \tag{21}$$

The following theorem hold.

Theorem 2 *The manifold A_n admits canonic almost geodesic mapping of type π_1 onto symmetric manifold $\bar{A}_n$ if and only if it contains a solution of a closed mixed system of Cauchy type equations in covariant derivative* (14)*,* (16)*,* (19) *and* (21) *in respect to unknown functions $\bar{R}^h_{ijk}(x)$, $P^h_{ij}(x)$ and $a_{ij}(x)$.*
General solution of the above system depends on no more than $1/2\, n(n^3 + 2n + 1)$ essential real parameters.

The systems (14), (16), (19) have only one solution for the initial conditions in point x_0

$$\bar{R}^h_{i(jk)}(x_0) = 0, \qquad P^h_{ij}(x_0), \qquad a_{ij}(x_0),$$

which has to satisfy the condition (21). From this follows the above number of essential real parameters.

CONCLUSION

In this paper we obtained the fundamental equations of geodesic mappings of spaces with affine connection onto Ricci symmetric manifolds and almost geodesic mappings of the first type of spaces with affine connection onto symmetric spaces. The fundamental equations have a closed Cauchy type form in covariant derivatives. We also set the number of essential real parameters for the general solution of such system.

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