Canonical almost geodesic mappings of type $\tilde{\pi}_1$ onto pseudo-Riemannian manifolds

V.E. Berezovski

Department of Mathematics, University of Uman,
Institutskaya 1, Uman, Ukraine
E-mail: berez.volod@rambler.ru

J. Mikeš and A. Vanžurová

Department of Algebra and Geometry, Faculty of Science, Palacký University
Tomkova 40, 779 00 Olomouc, Czech Republic
E-mail: josef.mikes@upol.cz, alena.vanzurova@upol.cz

Our aim is to examine almost geodesic mappings of affine manifolds and give necessary and sufficient conditions for existence of the so-called $\tilde{\pi}_1$ mappings (canonical almost geodesic mappings of type π according to Sinyukov) of a manifold endowed with a linear connection onto pseudo-Riemannian manifolds. The conditions take the form of a closed system of PDE's of first order of Cauchy type. Our result is a generalization of some previous theorems of N.S. Sinyukov.

Keywords: Linear connection, affine manifold, pseudo-Riemannian space, geodesic curve, almost geodesic curve, geodesic mapping, almost geodesic mapping, deformation tensor.

MS classification: 53B20, 53B30, 53B35.

1. Introduction

Unless otherwise specified, all objects under consideration are supposed to be differentiable of a sufficiently high class (mostly, differentiability of the class C^3 is sufficient).

Let $A_n = (M, \nabla)$ be an n-dimensional (C^k, C^∞ or C^ω) manifold endowed with a linear connection ∇ (an "affine manifold"). Let $c : I \to M$, $t \mapsto c(t)$ defined on an open interval $I \subset \mathbb{R}$ be a (C^k, or smooth) curve on M satisfying the regularity condition

$$c'(t) = dc(t)/dt \neq 0 \quad \text{for all} \quad t \in I.$$

Denote by ξ the corresponding (C^{k-1}, or smooth) tangent vector field along c ("velocity field"), $\xi(t) = (c(t), c'(t))$, $t \in I$, and let

$$\xi_1 = \nabla(\xi;\xi) = \nabla_\xi \xi, \qquad \xi_2 = \nabla^2(\xi;\xi,\xi) = \nabla_\xi \xi_1. \tag{1}$$

Geodesics $c(s)$, parametrized by canonical affine parameter (given up to affine transformations $s \mapsto as + b$), are characterized by $\nabla_\xi \xi = 0$ while unparametrized geodesic curves (i.e. arbitrarily parametrized, called also *pregeodesics* in the literature) can be characterized by the formula $\nabla_\xi \xi = \lambda \xi$ where $\lambda(t) : I \to \mathbb{R}$ is a real function.

Let $D = \operatorname{span}(X_1, X_2)$ (i.e. vector fields X_1, X_2 along c form a basis of D). Recall that D is parallel (along c) if and only if covariant derivatives along c of basis vector fields belong to the distribution (the property is independent on reparametrization of the curve).[1–3]

As a generalization of (an unparametrized) geodesic, let us introduce an *almost geodesic curve* as a curve c satisfying: there exists a two-dimensional (differentiable) distribution D parallel along c (relative to ∇) such that for any tangent vector of c, its parallel translation along c (to any other point) belongs to the distribution D.

Equivalently, c is almost geodesic if and only if there exist vector fields X_1, X_2 parallel along c (i.e. satisfying $\nabla_\xi X_i = a^j X_j$ for some differentiable functions $a_i^j(t) : I \to \mathbb{R}$) and differentiable real functions $b^i(t)$, $t \in I$ along c, such that $\xi = b^1 X_1 + b^2 X_2$ holds. For almost geodesic curves, the vector fields ξ_1 and ξ_2 belong to the corresponding distribution D. If the vector fields ξ and ξ_1 are independent at any point (and hence the (local) curve c is not a geodesic one), we can write $D = \operatorname{span}(\xi, \xi_1)$. So we get another equivalent characterization: a curve is almost geodesic if and only if $\xi_2 \in \operatorname{span}(\xi, \xi_1)$.

2. Almost geodesic mappings

Geodesic mappings of manifolds with linear connection (in short, affine manifolds) are (C^k)-diffeomorphisms characterized by the property that all geodesics are send onto (unparametrized in general) geodesic curves. The classification of geodesic mappings is more or less known.

Recall that even for Riemannian spaces, there is a lack of a nice simple criterion for decision when a given Riemannian space admits non-trivial geodesic mappings.

Let $A_n = (M, \nabla)$, $\bar{A}_n = (\bar{M}, \bar{\nabla})$ be n-dimensional affine manifolds, $n > 2$, endowed with torsion-free linear connections.

We may ask which (C^k-)diffeomorphisms of affine manifolds send almost geodesic curves onto almost geodesic ones again. The answer is: such mappings reduce to geodesic ones, since there are "too many" almost geodesic curves. It appears that the following definition is more acceptable.

We say that a (C^k-)diffeomorphism $f\colon M \to \bar{M}$ is *almost geodesic* if any geodesic curve of (M, ∇) is mapped under f onto an almost geodesic curve in $(\bar{M}, \bar{\nabla})$.

This concept of an almost geodesic mapping was introduced by N.S. Sinyukov,[1] and before by V.M. Chernyshenko,[4] from a rather different point of view. The theory of almost geodesic mappings was treated in Ref. 1–3.

Due to the fact that f is a diffeomorphism we can accept the useful convention that both linear connections ∇ and $\bar{\nabla}$ are in fact defined on the same underlying manifold M, so that we can consider their difference $P = \bar{\nabla} - \nabla$. That is, P is a $(1,2)$-tensor, called sometimes a *deformation tensor* of the given connections under f,[2] given by $\bar{\nabla}(X,Y) = \nabla(X,Y) + P(X,Y)$ for $X, Y \in \mathcal{X}(M)$. Since the connections are symmetric, P is also symmetric in X, Y. Of course, we identify objects on M with their corresponding objects on $\bar{M}$: a curve c on M identifies with its image $\bar{c} = f \circ c$, its tangent vector field $\xi(t)$ with the corresponding vector field $\bar{\xi}(t) = Tf(\xi(t))$ etc.

Besides the deformation tensor, we will use type $(1,3)$ tensor field, denoted by the same symbol P, introduced by

$$P(X,Y,Z) = \sum_{CS(X,Y,Z)} \nabla_Z P(X,Y) + P(P(X,Y),Z), \quad X,Y,Z \in \mathcal{X}(M),$$

where $\sum_{CS(\,,\,,\,)}$ means the cyclic sum on arguments in brackets (i.e. symmetrization without coefficients).

Almost geodesic diffeomorphisms $f\colon (M,\nabla) \to (M,\bar{\nabla})$ are characterized by the following condition on the type $(1,3)$ tensor P:

$$P(X_1,X_2,X_3) \wedge P(X_4,X_5) \wedge X_6 = 0, \quad X_i \in \mathcal{X}(M), \quad i = 1,\dots,6;$$

$X \wedge Y$ means the exterior product of X and Y, the decomposable bivector.

N.S. Sinyukov[1–3] distinguished three kinds of almost geodesic mappings, namely π_1, π_2, and π_3, characterized, respectively, by the conditions for the deformation tensor:

$$\pi_1\colon \nabla_X P(X,X) + P(P(X,X),X) = a(X,X)\cdot X + b(X)\cdot P(X,X),\ X \in \mathcal{X}(M),$$

where $a \in S^2(M)$ is a symmetric type $(0,2)$ tensor field and b is a 1-form;

$$\pi_2\colon P(X,X) = \psi(X)\cdot X + \varphi(X)\cdot F(X), \quad X \in \mathcal{X}(M),$$

where ψ and φ are 1-forms, and F is a type $(1,1)$ tensor field satisfying

$$\nabla_X F(X) + \varphi(X)\cdot F(F(X)) = \mu(X)\cdot X + \varrho(X)\cdot F(X), \quad X \in \mathcal{X}(M)$$

for some 1-forms μ, ϱ;

$$\pi_3\colon P(X,X) = \psi(X)\cdot X + a(X,X)\cdot Z, \quad X \in \mathcal{X}(M)$$

where ψ is a 1-form, $a \in S^2(M)$ is a symmetric bilinear form and $Z \in \mathcal{X}(M)$ is a vector field satisfying

$$\nabla_X Z = h\cdot X + \theta(X)\cdot Z$$

for some scalar function $h\colon M \to \mathbb{R}$ and some 1-form θ. Remark that the above classes are not disjoint.

3. Canonical almost geodesic mappings $\tilde{\pi}_1$

We are interested here in a particular subclass of π_1-mappings, the so-called $\tilde{\pi}_1$-*mappings*, or *canonical* almost geodesic mappings, distinguished by the condition $b = 0$. That is, $\tilde{\pi}_1$-mappings are just morphisms satisfying

$$\nabla_X P(X,X) + P(P(X,X),X) = a(X,X)\cdot X, \quad a \in S^2(M),\ X \in \mathcal{X}(M).$$

In local coordinates, the condition reads

$$P^h_{(ij,k)} = a_{(ij}\delta^h_{k)} - P^h_{\alpha(i}P^\alpha_{jk)}. \tag{2}$$

Here and after the comma "," denotes covariant derivative with respect to ∇, δ^h_i is the Kronecker delta, the round bracket denote the cyclic sum on indices.

Any geodesic mapping is a π_1-mapping (the characterizing condition can be checked), and any π_1-mapping can be written as a composition of a geodesic mapping followed by a $\tilde{\pi}_1$-mapping. So we can consider geodesic mappings as trivial almost geodesic mappings, and we will omit them in further considerations; they have been analysed in Ref. 5. It was proven by Sinyukov[2] that the basic partial differential equations (PDE's) of $\tilde{\pi}_1$-mappings of an affine manifold (M,∇) onto Ricci-symmetric ($\bar{\nabla}\bar{\text{R}}\text{ic} = 0$, the Ricci tensor is parallel) pseudo-Riemannian spaces $(\bar{M},\bar{g})$ (of arbitrary signature) can be transformed into (an equivalent) closed system of PDE's of first order of Cauchy type. Hence the solution (if it exists) depends on a finite set of parameters. Consequently, for an affine manifold with a symmetric connection admitting $\tilde{\pi}_1$-mappings onto Ricci-symmetric spaces, the set of all Ricci-symmetric spaces $(\bar{M},\bar{g})$ which can serve as images of the given affine manifold (M, ∇) under $\tilde{\pi}_1$-mappings is finite. The cardinality r of such a set is bounded by the number of free parameters.

On the other hand, geodesic mappings form a subclass among $\tilde{\pi}_1$-mappings (they obey the definition). Basic equations describing geodesic mappings of affine manifolds do not form a closed system of Cauchy type (the general solution depends on n arbitrary functions; if the given manifold admits geodesic mappings, the cardinality of the set of possible images is big). It follows that the conditions (2) describing $\tilde{\pi}_1$-mappings of affine manifolds, in general, cannot be transformed into a closed system of Cauchy type. But if we choose a suitable subclass of images and restrict ourselves (for the given manifold) only onto mappings with co-domain in the apropriate subclass we might succeed to get an equivalent closed system of Cauchy type. If this is the case then the given manifold admits either non (if the system is non-integrable) or a finite number of $\tilde{\pi}_1$-images in the given class.

Our aim is to analyse $\tilde{\pi}_1$-mappings of affine manifolds onto affine manifolds in general, and to use the reached results for examining $\tilde{\pi}_1$-mappings of affine manifolds onto (pseudo-)Riemannian spaces (in general, without any restrictive conditions onto Ricci tensor), which will generalize the above result by Sinyukov. In the rest, we will omit "pseudo".

All $\tilde{\pi}_1$-mappings $f\colon M \to M$ can be described by the following system of (differential) equations:[2,3]

$$3(\nabla_Z P(X,Y) + P(Z, P(X,Y))) =$$

$$\sum_{CS(X,Y)} (R(Y,Z)X - \bar{R}(Y,Z)X) + \sum_{CS(X,Y,Z)} a(X,Y)Z.$$

In the rest, we prefer to express our equalities in local coordinates (with respect to a map (U, φ) on M) since the invariant formulas are rather complicated. The above formula has the local expression

$$3(P^h_{ij,k} + P^h_{k\alpha} P^\alpha_{ij}) = R^h_{(ij)k} - \bar{R}^h_{(ij)k} + a_{(ij}\delta^h_{k)}, \tag{3}$$

where P^h_{ij}, a_{ij}, R^h_{ijk}, $\bar{R}^h_{ijk}$ are local components of tensors P, a, R, and $\bar{R}$.

Assuming (15) as a system of PDE's for functions P^h_{ij} on M, the corresponding integrability conditions read

$$\begin{aligned}\bar{R}^h_{(ij)[k,\ell]} &= R^h_{(ij)[k,\ell]} + \delta^h_{(i}a_{jk),\ell} - \delta^h_{(i}a_{j\ell),k} + 3(P^\alpha_{ij}\bar{R}^h_{\alpha k\ell} - P^h_{\alpha(j}R^\alpha_{i)k\ell}) \\ &\quad - P^h_{\alpha k}(R^\alpha_{(ij)\ell} - \bar{R}^\alpha_{(ij)\ell}\delta^\alpha_{(i}a_{j\ell)}) + P^h_{\alpha\ell}(R^\alpha_{(ij)k} - \bar{R}^\alpha_{(ij)k}\delta^\alpha_{(i}a_{jk)})\,.\end{aligned}$$

Passing from $\nabla\bar{R}$ to $\bar{\nabla}\bar{R}$ on the left hand side we get integrability conditions of the system (15) in the form

$$\bar{R}^h_{(ij)[k;\ell]} = \delta^h_{(i}a_{jk),\ell} - \delta^h_{(i}a_{j\ell),k} + \Theta^h_{ijk\ell}\,, \tag{4}$$

where we denoted

$$\Theta^h_{ijk\ell} = R^h_{(ij)[k,\ell]} + 3(P^\alpha_{ij}\bar{R}^h_{\alpha k\ell} - P^h_{\alpha(j}R^\alpha_{i)k\ell}) - P^h_{\alpha k}(R^\alpha_{(ij)\ell} + \delta^\alpha_{(i}a_{j\ell)})$$
$$+ P^h_{\alpha\ell}(R^\alpha_{(ij)k} + \delta^\alpha_{(i}a_{jk)}) - P^\alpha_{\ell(i}\bar{R}^h_{|\alpha|j)k} - P^\alpha_{\ell(i}\bar{R}^h_{j)\alpha k} + P^\alpha_{k(i}\bar{R}^h_{|\alpha|j)\ell} + P^\alpha_{k(i}\bar{R}^h_{j)\alpha\ell}\,,$$

where ";" denotes covariant derivative with respect to $\bar{\nabla}$.

If we apply covariant differentiation with respect to $\bar{\nabla}$ to the integrability conditions (17) of the system (15), and then pass from covariant derivation $\bar{\nabla}$ to ∇, we get

$$\bar{R}^h_{(ij)k;\ell m} - \bar{R}^h_{(ij)\ell;mk} = \delta^h_{(i}a_{jk),\ell m} - \delta^h_{(i}a_{j\ell),km} + T^h_{ijk\ell m}\,, \tag{5}$$

where we denoted

$$T^h_{ijk\ell m} = \bar{R}^h_{\alpha mk}\bar{R}^\alpha_{(ij)\ell} - \bar{R}^\alpha_{\ell mk}\bar{R}^h_{(ij)\alpha} - \bar{R}^\alpha_{jmk}\bar{R}^h_{(i\alpha)\ell} - \bar{R}^\alpha_{imk}\bar{R}^h_{(j\alpha)\ell}$$
$$- P^h_{m\alpha}\delta^\alpha_{(i}a_{jk),\ell} - P^\alpha_{mj}\delta^h_{(i}a_{\alpha k),\ell} - P^\alpha_{mi}\delta^h_{(\alpha}a_{jk),\ell} - P^\alpha_{mk}\delta^h_{(\alpha}a_{ij),\ell}$$
$$- P^\alpha_{ml}\delta^h_{(i}a_{jk),\alpha} - P^h_{m\alpha}\delta^\alpha_{(i}a_{j\ell),k} + P^\alpha_{mi}\delta^h_{(\alpha}a_{j\ell),k} + P^\alpha_{mj}\delta^h_{(i}a_{\alpha\ell),k}$$
$$+ P^\alpha_{mk}\delta^h_{(i}a_{j\ell),\alpha} - P^\alpha_{ml}\delta^h_{(i}a_{j\alpha),k} - \Theta^h_{ijk\ell,m} + P^h_{\alpha m}\Theta^\alpha_{ijk\ell} - P^\alpha_{mi}\Theta^h_{\alpha jk\ell}$$
$$- P^\alpha_{mj}\Theta^h_{i\alpha k\ell} - P^\alpha_{mk}\Theta^h_{ij\alpha\ell} - P^\alpha_{m\ell}\Theta^h_{ijk\alpha}\,.$$

Alternating (24) in ℓ, m yields

$$\begin{aligned}\bar{R}^h_{(ij)m;\ell k} - \bar{R}^h_{(ij)\ell;mk} &= \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(i}a_{j\ell),km} + T^h_{ijk[lm]}\\ &+ \bar{R}^h_{(i|\alpha k|}\bar{R}^\alpha_{j)m\ell} + \bar{R}^h_{(ij)\alpha}\bar{R}^\alpha_{km\ell} - \bar{R}^\alpha_{(ij)k}\bar{R}^h_{\alpha m\ell} + \bar{R}^h_{\alpha(i|k|}\bar{R}^\alpha_{j)m\ell}\\ &+ \delta^h_{(\alpha}a_{jk)}R^\alpha_{i\ell m} + \delta^h_{(\alpha}a_{ik)}R^\alpha_{j\ell m} + \delta^h_{(i}a_{j\alpha)}R^\alpha_{k\ell m} - \delta^h_{(i}a_{jk)}R^\alpha_{\alpha\ell m}\,.\end{aligned} \tag{6}$$

Using properties of the Riemannian tensor, we rewrite (26) as

$$\bar{R}^h_{im\ell;jk} + \bar{R}^h_{jm\ell;ik} = \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - N^h_{ijk\ell m}\,, \tag{7}$$

where the last term is

$$N^h_{ijk\ell m} = T^h_{ijk[\ell m]} + \bar{R}^\alpha_{im\ell}\bar{R}^h_{(\alpha j)k} + \bar{R}^\alpha_{jm\ell}\bar{R}^h_{(\alpha i)k} + \bar{R}^\alpha_{km\ell}\bar{R}^h_{(ij)\alpha}$$
$$- \bar{R}^h_{\alpha m\ell}\bar{R}^\alpha_{(ij)k} + \delta^h_{(\alpha}a_{jk)}R^\alpha_{i\ell m} + \delta^h_{(\alpha}a_{ik)}R^\alpha_{j\ell m} + \delta^h_{(\alpha}a_{ij)}R^\alpha_{k\ell m} - a_{(ij}R^h_{k)\ell m}.$$

Alternating (27) in j, k we get

$$\begin{aligned}\bar{R}^h_{jm\ell;ik} - \bar{R}^h_{km\ell;ij} &= \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(i}a_{k\ell),jm} + \delta^h_{(i}a_{km),j\ell}\\ &- N^h_{i[jk]\ell m} + \bar{R}^h_{\alpha m\ell}\bar{R}^\alpha_{ikj} + \bar{R}^h_{i\alpha\ell}\bar{R}^\alpha_{mkj} + \bar{R}^h_{im\alpha}\bar{R}^\alpha_{\ell kj} - \bar{R}^\alpha_{im\ell}\bar{R}^h_{\alpha kj}\,.\end{aligned} \tag{8}$$

Let us change mutually i and k in (27), and then use (29). We evaluate

$$\begin{aligned}2\bar{R}^h_{jm\ell;ik} &= \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(k}a_{jm),i\ell}\\ &+ \delta^h_{(i}a_{km),j\ell} - \delta^h_{(i}a_{k\ell),jm} + \delta^h_{(j\ell}a_{k),im} + \Omega^h_{ijk\ell m}\,,\end{aligned} \tag{9}$$

where we used the notation

$$\Omega^h_{ijk\ell m} = -N^h_{ijk\ell m} + N^h_{k[ij]k\ell m} - \bar{R}^h_{\alpha m\ell}\bar{R}^\alpha_{(kj)i} + \bar{R}^h_{j\alpha\ell}\bar{R}^\alpha_{mik} + \bar{R}^h_{jm\alpha}\bar{R}^\alpha_{\ell ik}$$
$$-\bar{R}^h_{\alpha i(j}\bar{R}^\alpha_{k)m\ell} + \bar{R}^h_{j\alpha\ell}\bar{R}^\alpha_{mik} + \bar{R}^h_{jm\alpha}\bar{R}^\alpha_{\ell ik} - \bar{R}^h_{\alpha m\ell}\bar{R}^\alpha_{ikj} - \bar{R}^h_{i\alpha\ell}\bar{R}^\alpha_{mkj} + \bar{R}^\alpha_{im[\ell}\bar{R}^h_{\alpha]kj}.$$

On the left side of (30), let us pass from covariant derivation $\bar{\nabla}$ to ∇:

$$2\bar{R}^h_{jm\ell,ik} = \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(k}a_{jm),i\ell} + \delta^h_{(i}a_{km),j\ell} - \delta^h_{(i}a_{k\ell),jm} - \delta^h_{(k}a_{j\ell),im} + S^h_{ijk\ell m}\,, \tag{10}$$

where

$$\begin{aligned} S^h_{ijk\ell m} = \Omega^h_{ijk\ell m} - 2\,[&\bar{R}^\alpha_{jm\ell,i}P^h_{\ell k} - \bar{R}^h_{\alpha m\ell,i}P^\alpha_{jk} - \bar{R}^h_{j\alpha\ell,i}P^\alpha_{mk} \\ &- \bar{R}^h_{jm\alpha,i}P^\alpha_{\ell k} - \bar{R}^h_{jm\ell,\alpha}P^\alpha_{ik} \\ &+ (\bar{R}^\alpha_{jm\ell}P^\beta_{\alpha i} - \bar{R}^h_{\alpha m\ell}P^\alpha_{ij} - \bar{R}^h_{j\alpha\ell}P^\alpha_{im} - \bar{R}^h_{jm\alpha}P^\alpha_{i\ell})P^h_{\beta k} \\ &- (\bar{R}^\alpha_{jm\ell}P^h_{\alpha\beta} - \bar{R}^h_{\alpha m\ell}P^\alpha_{\beta j} - \bar{R}^h_{j\alpha\ell}P^\alpha_{\beta m} - \bar{R}^h_{jm\alpha}P^\alpha_{\beta\ell})P^\beta_{ik} \\ &- (\bar{R}^\alpha_{\beta m\ell}P^h_{\alpha i} - \bar{R}^h_{\alpha m\ell}P^\alpha_{\beta i} - \bar{R}^h_{\beta\alpha\ell}P^\alpha_{im} - \bar{R}^h_{\beta m\alpha}P^\alpha_{i\ell})P^\beta_{jk} \\ &- (\bar{R}^\alpha_{j\beta\ell}P^h_{\alpha i} - \bar{R}^h_{\alpha\beta\ell}P^\alpha_{ji} - \bar{R}^h_{j\alpha\ell}P^\alpha_{\beta i} - \bar{R}^h_{j\beta\alpha}P^\alpha_{i\ell})P^\beta_{km} \\ &- (\bar{R}^\alpha_{jm\beta}P^h_{\alpha i} - \bar{R}^h_{\alpha m\beta}P^\alpha_{ji} - \bar{R}^h_{j\alpha\beta}P^\alpha_{mi} - \bar{R}^h_{jm\alpha}P^\alpha_{\beta i})P^\beta_{k\ell}]\,. \end{aligned} \tag{11}$$

Let there exist a $\tilde{\pi}_1$-mapping of an affine manifold $A_n = (M, \nabla)$ onto a Riemannian manifold $\bar{V}_n = (M, \bar{g})$ where $\bar{g} \in T^0_2 M$ is a metric tensor with components $\bar{g}_{ij}$. Recall that the Riemannian tensor $\bar{R}_{hijk} = \bar{R}^\alpha_{ijk}\bar{g}_{\alpha h}$ of type $(0,4)$ satisfies

$$\bar{R}_{hijk} + \bar{R}_{ihjk} = 0. \tag{12}$$

In (30), let us apply the metric tensor $\bar{g}_{h\beta}$ and then use symmetrization with respect to h and j. According to (12) we get

$$\begin{aligned} &\bar{g}_{ih}(a_{m[k,j]l} + a_{l[j,k]m}) + \bar{g}_{ij}(a_{m[k,h]l} + a_{l[h,k]m}) + \bar{g}_{kh}(a_{m[i,j]l} \\ &+ a_{l[j,i]}) + \bar{g}_{kj}(a_{m[i,h]l} + a_{l[h,i]m}) + \bar{g}_{mh}(a_{k[i,j]l} - a_{ij,kl}) \\ &+ \bar{g}_{mj}(a_{k[i,h]l} - a_{ih,kl}) + \bar{g}_{lj}(a_{kh,il} - a_{i(h,k)m}) \\ &+ 2\bar{g}_{jh}(a_{k(l,i)m} - a_{m(i,k)l}) + \bar{g}_{lh}(a_{k[j,i]m} - a_{ij,km}) = -\Omega^\alpha_{i(j|klm}\bar{g}_{\alpha|h)}. \end{aligned} \tag{13}$$

Contraction of the last formula with the dual tensor $\bar{g}^{jh}$ ($\|\bar{g}^{ij}\| = \|\bar{g}_{ij}\|^{-1}$) gives

$$a_{kl,im} - a_{im,kl} - a_{km,il} + a_{il,km} = -\frac{2}{n+1}\Omega^\alpha_{i\alpha klm}\,. \tag{14}$$

Let us symmetrize the above formula in k and l. From (14) we get

$$\begin{aligned} 2a_{kl,im} - 2a_{im,kl} = 2a_{\alpha m}R^\alpha_{lik} + a_{\alpha i}R^\alpha_{mlk} + a_{\alpha k}R^\alpha_{mil} + a_{\alpha l}R^\alpha_{mik} \\ + \frac{2}{n+1}(\Omega^\alpha_{l\alpha kim} - \Omega^\alpha_{i\alpha(kl)m}). \end{aligned} \tag{15}$$

Using (14) and (15) the equation (13) reads

$$\begin{aligned} &2\bar g_{ih}(a_{km,jl}-a_{jm,kl})+2\bar g_{ij}(a_{km,hl}-a_{hm,kl})+2\bar g_{kh}(a_{im,jl}-a_{jm,il})\\ &\quad+2\bar g_{kj}(a_{im,hl}-a_{hm,il})+\bar g_{mk}(a_{ki,jl}-a_{kj,il}-a_{ij,kl})\\ &\quad+\bar g_{mj}(a_{ki,hl}-a_{kh,il}-a_{ih,kl})+\bar g_{lj}(a_{kh,im}-a_{i(h,k)m})\\ &\quad+\bar g_{lh}(a_{kj,im}-a_{i(k,j)m})=C_{ijkhl}, \end{aligned} \tag{16}$$

where

$$\begin{aligned} C_{ijkhl}&=-\Omega^{\alpha}_{i(j|klm}\bar g_{\alpha|h)}+\frac{2}{n+1}\Omega^{\alpha}_{i\alpha klm}\bar g_{jh}-\bar g_{kh}a_{\alpha l}R^{\alpha}_{mij}\\ &+\bar g_{ih}\Big(\frac{2}{n+1}\Omega^{\alpha}_{m\alpha ljk}-a_{\alpha k}R^{\alpha}_{(ml)j}-a_{\alpha j}R^{\alpha}_{(l|k|m)}-a_{\alpha m}R^{\alpha}_{lkj}-a_{\alpha l}R^{\alpha}_{mkj}\Big)\\ &+\bar g_{ij}\Big(\frac{2}{n+1}\Omega^{\alpha}_{m\alpha lhk}-a_{\alpha k}R^{\alpha}_{(ml)h}-a_{\alpha h}R^{\alpha}_{(l|k|m)}-a_{\alpha m}R^{\alpha}_{lkh}-a_{\alpha l}R^{\alpha}_{mkh}\Big)\\ &+\bar g_{kh}\Big(\frac{2}{n+1}\Omega^{\alpha}_{m\alpha lji}-a_{\alpha i}R^{\alpha}_{(ml)j}-a_{\alpha j}R^{\alpha}_{(l|i|m)}-a_{\alpha m}R^{\alpha}_{lij}+a_{\alpha l}R^{\alpha}_{mij}\Big)\\ &+\bar g_{kj}\Big(\frac{2}{n+1}\Omega^{\alpha}_{m\alpha lhi}-a_{\alpha i}R^{\alpha}_{(ml)h}-a_{\alpha h}R^{\alpha}_{(l|i|m)}-a_{\alpha m}R^{\alpha}_{lih}+a_{\alpha l}R^{\alpha}_{mih}\Big). \end{aligned}$$

If we contract (16) with the dual $\bar g^{ij}$ of the metric tensor, use (15) and the Ricci identity we get

$$a_{km,hl}-a_{kl,hm}=\tfrac{1}{2(n+3)}(\bar g_{hm}\mu_{kl}-\bar g_{hl}\mu_{km})+B_{kmhl}, \tag{17}$$

where $\mu_{km}=a_{\alpha\beta,km}\bar g^{\alpha\beta}$, and

$$\begin{aligned} B_{kmhl}&=C_{\alpha\beta kmhl}\bar g^{\alpha\beta}+3a_{m\alpha}R^{\alpha}_{lhk}+\frac{3}{2}(a_{h\alpha}R^{\alpha}_{mkl}+a_{k\alpha}R^{\alpha}_{mhl}+a_{l\alpha}R^{\alpha}_{mhk})\\ &+\frac{3}{n+1}(\Omega^{\alpha}_{l\alpha khm}-\Omega^{\alpha}_{h\alpha(kl)m})-\frac{1}{2}(a_{m\alpha}R^{\alpha}_{lkm}+a_{k\alpha}R^{\alpha}_{mhl}+a_{h\alpha}R^{\alpha}_{mkl}+a_{l\alpha}R^{\alpha}_{mkh})\\ &-\frac{1}{n+1}(\Omega^{\alpha}_{l\alpha hkm}-\Omega^{\alpha}_{k\alpha(hl)m})-a_{\alpha(h}R^{\alpha}_{k)lm}+\frac{1}{2}(a_{k\alpha}R^{\alpha}_{lmh}+a_{h\alpha}R^{\alpha}_{lkm}+a_{m\alpha}R^{\alpha}_{lkh}). \end{aligned}$$

Now contract (16) with $\bar g^{ih}$. According to (17) we get

$$\bar g_{kl}\mu_{jm}-\bar g_{jl}\mu_{km}+\bar g_{km}\mu_{jl}-\bar g_{jkm}\mu_{kl}=\frac{n+3}{n+1}\,C_{kljm}, \tag{18}$$

where

$$\begin{aligned} C_{kljm}&=C_{\alpha jkl(m|\beta|l)}\bar g^{\alpha\beta}-2(n+1)(B_{k(ml)j}-a_{\alpha(l}R^{\alpha}_{m)jk}\\ &\quad+a_{j\alpha}R^{\alpha}_{(m|k|l)}+a_{k\alpha}R^{\alpha}_{(lm)j}). \end{aligned}$$

Contracting (18) with $\bar g^{k\ell}$ and using the notation $K=\mu_{\alpha\beta}\bar g^{\alpha\beta}$ we obtain components of the tensor μ:

$$\mu_{jm}=\frac{1}{n}K\bar g_{jm}+\frac{n+3}{n(n+1)}C_{\alpha\beta jm}\bar g^{\alpha\beta}. \tag{19}$$

Using (19) we can rewrite (17) in the form

$$a_{km,hl} - a_{hm,kl} = \frac{K}{2n(n+3)} (\bar{g}_{mh}\bar{g}_{kl} - \bar{g}_{lh}\bar{g}_{km}) + A_{kmhl}, \tag{20}$$

where

$$A_{kmhl} = B_{kmhl} + \frac{1}{2n(n+1)} (\bar{g}_{hm} C_{\alpha\beta kl}\bar{g}^{\alpha\beta} - \bar{g}_{hl} C_{\alpha\beta km}\bar{g}^{\alpha\beta}).$$

Combining (16) and (20) we get

$$\begin{aligned} &\bar{g}_{jl}a_{ih,km} + \bar{g}_{hl}a_{ij,km} - \bar{g}_{jm}a_{ih,kl} - \bar{g}_{hm}a_{ij,kl} = \\ &\quad - \frac{K}{n(n+3)} (\bar{g}_{ih}\bar{g}_{kl}\bar{g}_{jm} - \bar{g}_{ih}\bar{g}_{km}\bar{g}_{jl} + \bar{g}_{ij}\bar{g}_{kl}\bar{g}_{hm} - \bar{g}_{ij}\bar{g}_{km}\bar{g}_{hl} \\ &\quad + 3\bar{g}_{kh}\bar{g}_{il}\bar{g}_{jm} - 3\bar{g}_{kh}\bar{g}_{jl}\bar{g}_{im} + 3\bar{g}_{kj}\bar{g}_{il}\bar{g}_{hm} - 3\bar{g}_{lh}\bar{g}_{jk}\bar{g}_{im}) + A_{ijkmhl}, \end{aligned} \tag{21}$$

where we have denoted

$$A_{ijkmhl} = C_{ijkmhl} - 2(\bar{g}_{i(h}A_{|km|j)l} + \bar{g}_{k(h}A_{|im|jl)} - \bar{g}_{m(h}A_{|ki|j)l} - \bar{g}_{l(h}A_{|k|j)im)}).$$

Finally, symetrization of (21) in indices i, j, followed by contraction with $\bar{g}^{\ell h}$, anables us to express second covariant derivatives of the tensor a,

$$a_{ij,km} = \frac{K}{n(n+3)} (\bar{g}_{ij}\bar{g}_{km} + 3\bar{g}_{k(j}\bar{g}_{i)m}) + A_{(ij)km\alpha\beta}\bar{g}^{\alpha\beta}. \tag{22}$$

Now we can consider (22) as the system of PDE's (of first order) of Cauchy type relative to the tensor ∇a (i.e. in $a_{ij,k}$), find the integrability conditions and contract them with $\bar{g}^{ij}$ and $\bar{g}^{km}$, respectively. We calculate ∇K,

$$K_{,\beta} = \frac{n(n+3)}{n^2+5n-6} A_\beta, \tag{23}$$

where we denoted

$$\begin{aligned} A_\varrho = \Big[&a_{\alpha(j,|k}R^\alpha_{i)m\varrho} + a_{ij,\alpha}R^\alpha_{km\varrho} - \frac{K}{n(n+3)} (\bar{g}_{ij,[\varrho}\bar{g}_{m]k} + \bar{g}_{ij}\bar{g}_{k[m,\varrho} \\ &+ 3\bar{g}_{kj,[\varrho}\bar{g}_{m]i} + 3\bar{g}_{kj}\bar{g}_{i[m,\varrho]} + 3\bar{g}_{ki,[\varrho}\bar{g}_{m]j} + 3\bar{g}_{ki}\bar{g}_{j[m,\varrho]}) \\ &+ A_{(ij)k[m|\alpha\beta|,\varrho]}\bar{g}^{\alpha\beta} + A_{(ij)k[m|\alpha\beta|}\bar{g}^{\alpha\beta}{}_{,\varrho]}\Big] \bar{g}^{ij}\bar{g}^{km}. \end{aligned}$$

We use $\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + P^h_{ij}$ and get

$$\bar{g}_{ij,k} = P^\alpha_{ik}\bar{g}_{\alpha j} + P^\alpha_{jk}\bar{g}_{\alpha i}. \tag{24}$$

Assume the tensors ∇a and $\nabla \bar{R}$, and denote their components by $a_{ijk} := a_{ij,k}$ and $\bar{R}^h_{ijk\ell} := \bar{R}^h_{ijk,\ell}$, respectively. Then (32) and (22) take the form

$$\begin{aligned} 2R^h_{jmli,k} = &\delta^h_{(i}a_{jl)k,m} - \delta^h_{(i}a_{jm)k,l} + \delta^h_{(k}a_{jl)i,m} - \delta^h_{(k}a_{jm)i,l} \\ &+ \delta^h_{(i}a_{km)j,l} - \delta^h_{(i}a_{kl)j,m} + S^h_{ijklm}, \end{aligned} \tag{25}$$

$$a_{ijk,m} = \frac{K}{n(n+3)}(\bar{g}_{ij}\bar{g}_{km} + 3\bar{g}_{k(j}\bar{g}_{i)m}) + A_{(ij)km\alpha\beta}\bar{g}^{\alpha\beta}, \tag{26}$$

where covariant derivatives of the tensor a_{ijk} in (25) are supposed to be expressed according to (26), the tensor S was introduced componentwise in (33).

The formulas (15), (23)–(26) represent a closed system of Cauchy type for unknown functions

$$\bar{g}_{ij}(x),\ P^h_{ij}(x),\ a_{ij}(x),\ a_{ijk}(x),\ K(x),\ \bar{R}^h_{ijk}(x),\ R^h_{ijkl}(x), \tag{27}$$

which, moreover, must satisfy a finite set of algebraic conditions

$$\bar{g}_{[ij]} = P^h_{[ij]} = a_{[ij]} = a_{[ij]k} = \bar{R}^h_{i(jk)} = R^h_{i(jk)l} = 0,\ det\|\bar{g}_{ij}(x)\| \neq 0. \tag{28}$$

So we have proven:

Theorem 3.1. *The given affine manifold $A_n = (M, \nabla)$ admits $\tilde{\pi}_1$-mappings (i.e. canonical almost geodesic mappings of type π_1) onto Riemannian spaces $\bar{V}_n = (M, \bar{g})$ if and only if there exists solution of the mixed system of Cauchy type* (15), *(23)-(26), (28) for functions (27).*

As a consequence of the additional algebraic conditions, we get an upper boundary for the number r of possible solutions:

Corollary 3.1. *The family of all Riemannian manifolds $\bar{V}_n$ which can serve as images of the given affine manifold $A_n = (M, \nabla)$, depends on at most*

$$\frac{1}{2}n^2(n^2-1) + n(n+1)^2 + 1$$

parameters.

The above Theorem generalizes the result of Sinyukov[3] already mentioned as well as his results on geodesic mappings of Riemannian spaces.

Acknowledgments

Supported by grant No. 201/05/2707 of The Grant Agency of Czech Republic and by the Council of Czech Government MSM 6198959214.

References

1. N.S. Sinyukov, Almost geodesic mappings of affine-connected and Riemannian spaces. *DAN SSSR* **151** (4) (1963) 781–782 (in Russian).

2. N.S. Sinyukov, *Geodesic mappings of Riemannian spaces* (Nauka, Moscow, 1979).
3. N.S. Sinyukov, Almost geodesic mappings of affine-connected and Riemannian spaces. (Russian) *Itogi Nauki Tekh.*, Ser. Probl. Geom. **13** (1982) 3–26; Transl. in *J. Sov. Math.* **25** (1984) 1235–1249.
4. V.M. Chernyshenko, Affine-connected spaces with a correspondent complex of geodesics. *Collection of Works of Mech.-Math. Chair of Dnepropetrovsk Univ.* **6** (1961) 105–118.
5. V. Berezovsky and J. Mikeš, On a classification of almost geodesic mappings of affine connection spaces. *Acta Univ. Palacki. Olomuc., Fac. Rerum Nat., Math.* **35** (1996) 21–24.