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SHORT COMMUNICATIONS

Conformal Mappings of Riemannian Spaces onto Ricci Symmetric Spaces

V. E. Berezovskii[1*], I. Hinterleitner[2], N. I. Guseva[3***], and J. Mikeš[4****]**

[1] *Uman National University of Horticulture, Ukraine*
[2] *Brno University of Technology, Brno, Czechia*
[3] *Moscow State Pedagogical University, Moscow, Russia*
[4] *Palacky University, Olomouc, Czechia*

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1. INTRODUCTION

Conformal mappings of n-dimensional Riemannian spaces V_n have been considered by many authors. They find applications in general relativity theory (see, e.g., [1]–[8]).

In what follows, we assume the signature of the metrics on the spaces V_n under consideration to be arbitrary; i.e., each V_n may be a Riemannian space proper or a pseudo-Riemannian space.

Brinkmann reduced the question of whether a Riemannian space V_n, $n \geq 3$, can be conformally mapped onto an Einstein space $\overline{V}_n$ to the problem of solvability of a nonlinear system of Cauchy-type differential equations with covariant derivatives for $n+1$ unknown functions [2]. This problem was considered in detail by Petrov in his monograph [4].

In [9] and [10] (see also [6], [7], and [11]), the basic equations of the mappings in question were reduced to a linear system of Cauchy-type differential equations with covariant derivatives, which has made it possible to estimate the number r of arbitrary parameters determining a general solution of the problem. In other words, the mobility degree of Riemannian spaces with respect to conformal mappings onto Einstein spaces was found.

In [12], the first lacuna in the distribution of mobility degrees of Riemannian spaces with respect to conformal mappings onto Einstein spaces was estimated. As is known [10], the spaces with maximum mobility degree $r = n+2$ are precisely the conformally flat spaces.

A tensor characterization of nonconformally flat Riemannian spaces for which $r = n-1$ has been obtained. Thus, a sharp bound for the first lacuna in the distribution of mobility degrees of Riemannian spaces with respect to conformal mappings onto Einstein spaces was found, and the maximally mobile nonconformally flat Riemannian spaces with given degrees of mobility was described.

In these studies, the geometric objects under consideration were assumed to be of sufficiently high degree of smoothness.

In [13], minimal conditions on the differentiability of geometric objects under conformal mappings of Riemannian spaces V_n onto Einstein spaces were investigated. A system of equations determining these mappings was found in the form of a closed linear Cauchy-type system with covariant derivatives under minimum requirements on the differentiability of the metrics of the conformally corresponding spaces under consideration.

[*] E-mail: Berez.Volod@rambler.ru
[**] E-mail: Hinterleitner@fce.vutbr.cz
[***] E-mail: Ngus12@mail.ru
[****] E-mail: Josef.Mikes@upol.cz

In this paper, we consider conformal mappings of Riemannian spaces V_n onto Ricci symmetric Riemannian spaces. Note that Ricci symmetric spaces are characterized by the covariant constancy of the Ricci tensor; thereby, they are a natural generalization of Einstein spaces.

This opens a new natural direction of development for the problem on conformal mappings onto Einstein spaces stated above. In what follows, we obtain the basic equations of the problem about conformal mappings onto Ricci symmetric spaces in the form of a closed Cauchy-type system with covariant derivatives. We determine the number of essential parameters on which a general solution of this (nonlinear) system depends.

To conclude the introduction, we recall that geodesic mappings of Einstein and Ricci symmetric spaces were studied by Mikeš in [14]–[16]; see also [6], [7].

2. BASIC NOTIONS OF THE THEORY OF CONFORMAL MAPPINGS

First, we recall the basic notions of the theory of conformal mappings of Riemannian spaces; they can be found in [1], [4]–[7].

Consider a mapping f of a Riemannian space V_n with metric tensor g onto a Riemannian space $\overline{V}_n$ with metric tensor $\overline{g}$.

Suppose that the Riemannian spaces V_n and $\overline{V}_n$ are equipped with a coordinate system consistent with the mapping; we denote these coordinates by $x = (x^1, x^2, \dots, x^n)$ We assume that $n > 2$ and all functions under consideration are sufficiently smooth.

A mapping $f\colon V_n \to \overline{V}_n$ is said to be *conformal* if, in coordinates x consistent with f, the metric tensors g and $\overline{g}$ are proportional and their components satisfy the condition

$$\overline{g}_{ij}(x) = e^{2\psi(x)} g_{ij}(x), \tag{1}$$

where $\psi(x)$ is a function.

It follows from (1) that conformal mappings preserve the angles between tangent vectors to curves. This property completely characterizes conformal mappings.

Condition (1) implies the following dependence between the Christoffel symbols of the second kind of the spaces V_n and $\overline{V}_n$:

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi_i \delta^h_j + \psi_j \delta^h_i - \psi^h(x) g_{ij}(x), \tag{2}$$

where $\psi_i = \partial\psi/\partial x^i$ is the gradient vector, $\psi^h = g^{h\alpha}\psi_\alpha$, the g^{ij} are the components of the matrix inverse to that with components g_{ij}, and the δ^h_i are Kronecker deltas.

A conformal mapping is called a *homothety* if the function $\psi(x)$ is constant, i.e., if we have $\overline{g}_{ij}(x) = \mathrm{const}\, g_{ij}(x)$. This condition is equivalent to $\psi_i(x) = 0$; therefore, such a mapping is affine.

It is well known that the tensors of Riemannian spaces V_n and $\overline{V}_n$ related by a conformal mapping satisfy the conditions [1], [4]–[7]

$$\overline{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik} + g_{ij}\psi^h_k - g_{ik}\psi^h_j, \qquad \text{where} \quad \psi_{ij} = \psi_{i,j} - \psi_i\psi_j, \quad \psi^h_i = g^{h\alpha}\psi_{\alpha i}. \tag{3}$$

Convolving (3) over h and k, we see that the Ricci tensors of the spaces V_n and $\overline{V}_n$ are related by

$$\overline{R}_{ij} = R_{ij} - (n-2)\psi_{ij} + \mu g_{ij}, \tag{4}$$

where μ is a function.