

ALMOST GEODESIC MAPPINGS ONTO GENERALIZED RICCI-SYMMETRIC MANIFOLDS

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ABSTRACT. Our aim is to continue investigations concerning existence of almost geodesic mappings of manifolds with linear connection. We deduce necessary and sufficient conditions for existence of the so-called canonical almost geodesic mappings of type π of a manifold endowed with a linear connection onto generalized Ricci-symmetric manifolds. Our result is a generalization of some previous results by N. S. Sinyukov.

1. INTRODUCTION

First let us recall the main concepts and terminology. Let (M, ∇) be a smooth (C^∞) n -dimensional manifold endowed with a linear connection ∇ . Let TM denote the tangent bundle of M , let $p_M: TM \rightarrow M$ be the natural projection, and let $\Lambda^2 TM$ denote the associated vector bundle of bivectors. $\mathcal{X}(M)$ denotes the $\mathcal{F}(M)$ -module of vector fields on M over the ring $\mathcal{F}(M)$ of smooth functions on M . If $f: M \rightarrow \bar{M}$ is a diffeomorphism then $Tf: TM \rightarrow T\bar{M}$ is the corresponding tangent mapping, or differential, $Tf = f_*$. Unless otherwise specified, all objects under consideration are supposed to be differentiable of a sufficiently high class.

1.1. Vector fields and distributions parallel along a curve. Recall that an n -dimensional distribution on an open neighborhood $U \subseteq M$ ($\dim M = m \geq n$) is a map $D: U \rightarrow TU$, $U \ni x \mapsto D_x \subseteq T_x M$, and D is called differentiable of the class C^k if it admits a local C^k -basis around any point. In short, $D = \text{span}(X_1, \dots, X_n)$ on U . A C^k -vector field X (on U) belongs to D , $X \in D$, if $X_x \in D_x$ for all $x \in U$.

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Let $c: I \rightarrow M$, $t \mapsto c(t)$, with $I \subset \mathbb{R}$ being an open interval, denote a $(C^k$ -, or smooth) curve on M . Let ξ denote the corresponding $(C^{k-1}$ -, or smooth) tangent (“velocity”) vector field along c , $\xi(t) = \left(c(t), \frac{dc(t)}{dt}\right)$, $t \in I$. In the following, we will consider only those curves which are *regular* in the sense that the tangent vector field ξ along c does not vanish on the definition domain I , that is, $c'(t) = \frac{dc(t)}{dt} \neq 0$ for all $t \in I$. Besides the velocity field ξ , let us introduce vector fields ξ_1, ξ_2 , associated to a curve c , by the formula

$$(1) \quad \xi_1 = \nabla_\xi \xi, \quad \xi_2 = \nabla_\xi \xi_1$$

Under a $(C^k$ -)vector field along c we mean a $(C^k$ -)mapping $Y: I \rightarrow TM$ such that $p_M \circ Y = c$, that is, $Y(t) \in T_{c(t)}M$ for any $t \in I$. Similarly, a differentiable n -distribution along c can be introduced as a span of an n -tuple of (differentiable) vector fields along c . The velocity field $\xi(t)$ generates a one-dimensional distribution Ξ along c . Remark that any differentiable vector field (differentiable distribution, respectively) along c can be extended into a differentiable vector field (distribution) on some neighborhood U of $c(I)$.

Denote by $\tau_{c(t_0), c(t)}$ the parallel transport along c relative to ∇ from $c(t_0)$ to $c(t)$. A vector field Y along c on (M, ∇) is called *parallel* along c relative to the given connection if $\nabla_\xi Y = 0$. A distribution D (defined along c , or on some open neighborhood of $c(I)$) is called *parallel along c* if for any $t_0 \in I$ and any vector $X_0 \in D_{c(t_0)}$, the image of X_0 under the parallel translation τ along c (from the point $c(t_0)$ to $c(t)$) belongs to D , $\tau_{c(t_0), c(t)} X_0 \in D_{c(t)}$ for all $t \in I$. A distribution D parallel along c admits a (local) basis parallel along c ; parallelism along c is independent on reparametrizations of the path.

Lemma 1. *Let D be a two-dimensional distribution along c . Let X_1, X_2 be vector fields along c which form a basis of D ; $D = \text{span}(X_1, X_2)$. Then the necessary and sufficient condition for D to be parallel along c may be expressed as follows: there are real function $a_i^j: I \rightarrow \mathbb{R}$ of the parameter t such that*

$$(2) \quad \nabla_\xi X_i = a_i^j X_j, \quad i, j \in \{1, 2\}$$

hold (covariant derivatives along c of basis vector fields belong to the distribution, [5, p. 4].

1.2. Almost geodesic curves. Let (M, ∇) be a smooth manifold endowed with a linear connection. Let $c: I \rightarrow M$ be a smooth regular curve on M . Recall that c is called a *geodesic curve* (in short, g.c.), under a general parametrization, if for any initial value $t_0 \in I$ of the parameter, the vector field $\tau_{c(t_0), c(t)}(\xi(t_0))$ along c arising from images of $\xi(t_0)$ under the parallel propagation along c , belongs to the 1-dimensional distribution $\Xi = \text{span}(\xi)$ along c (generated by the velocity field). Hence the vectors $\xi_1(t)$ and $\xi(t)$ are collinear for any $t \in I$ if and only if c is a geodesic curve. Equivalently, c is a g. c. if and only if ξ is recurrent along

c which means: there is a real function $\lambda(t): I \rightarrow \mathbb{R}$ such that the formula

$$(3) \quad \nabla_{\xi(t)} \xi(t) = \lambda(t) \xi(t)$$

holds. If the curve is parametrized by canonical (affine) parameter, the condition (3) for geodesic curves take the usual form $\nabla_{\xi(s)} \xi(s) = 0$ for $s \in I$, and we speak about geodesics.

Geodesic curves can be naturally generalized as follows. According to [5], we call c *almost geodesic* if there is a 2-dimensional (differentiable) distribution D (along c , or on some neighborhood of $c(I)$) parallel along c relative to ∇ such that the tangent vector field ξ belongs to D (or Ξ is a subdistribution of D), $\xi(t) \in D_{c(t)}$ for $t \in I$. Equivalently, c is almost geodesic if and only if there exist vector fields X_1, X_2 along c satisfying (2), i.e. parallel, and (differentiable) real functions $b^i(t)$, $t \in \mathbb{R}$, defined along c , such that $\xi = b^1 X_1 + b^2 X_2$ holds. Obviously, geodesic curves, particularly geodesics, can serve as examples of almost geodesic curves.

For almost geodesic curves, ξ_1 and ξ_2 from (1) belong to the corresponding distribution D . If the vector fields ξ and ξ_1 are independent at any point (the (local) curve c is not a geodesic one), then $D = \text{span}(\xi, \xi_1)$. So we can easily check that another equivalent characterization is:

Lemma 2. *A curve is almost geodesic if and only if $\xi_2 \in \text{span}(\xi, \xi_1)$.*

2. ALMOST GEODESIC MAPPINGS

The concept of an almost geodesic mapping was introduced by V. M. Chernyshenko [3], and later on by N. S. Sinyukov, from a rather different point of view, [5, 6, 7, 8]. The theory of almost geodesic mappings in a developed form can be found in [5, 6, 7, 8].

Let (M, ∇) , $(\bar{M}, \bar{\nabla})$ be smooth n -dimensional manifolds, $n > 2$, endowed with torsion-free linear connection.

Definition. [5, 6, 7, 8] A diffeomorphism $f: M \rightarrow \bar{M}$ is called *almost geodesic* if any geodesic curve of (M, ∇) is mapped under f onto an almost geodesic curve in $(\bar{M}, \bar{\nabla})$.

Conventions. From now on, all connections under consideration are torsion-free ($\equiv$ symmetric). If $f: M \rightarrow \bar{M}$ is a diffeomorphism we always suppose that the connections ∇ and $\bar{\nabla}$ are defined on the same manifold M , and we may in fact assume diffeomorphisms $f: (M, \nabla) \rightarrow (M, \bar{\nabla})$, which is more convenient from the technical reasons: we can make use of the well-known fact that two linear connections ∇ and $\bar{\nabla}$ on the same manifold M always differ up to a $(1, 2)$ -tensor field P ,

$$(4) \quad \bar{\nabla}(X, Y) = \nabla(X, Y) + P(X, Y), \quad X, Y \in \mathcal{X}(M),$$

and if the connections are symmetric, then P is also symmetric in X, Y . Moreover, we always identify a given curve c with its image $\bar{c} = f \circ c$, similarly we

identify the tangent vector function $\xi(t)$ with the corresponding vector function $\bar{\xi}(t) = Tf(\xi(t))$. Given a diffeomorphism $f: (M, \nabla) \rightarrow (M, \bar{\nabla})$ then P determined by (4) will be called here the *deformation tensor* of the given connections under f ([6]). For a deformation tensor P (of type $(1, 2)$), let us introduce a new tensor field (of type $(1, 3)$, denoted by the same symbol) by

$$P(X, Y, Z) = \sum_{CS(X, Y, Z)} \nabla_Z P(X, Y) + P(P(X, Y), Z), \quad X, Y, Z \in \mathcal{X}(M)$$

where $\sum_{CS(., ., .)}$ means the cyclic sum on arguments in brackets (i. e. symmetrization without coefficients). Let $X \wedge Y$ means a decomposable bivector, an exterior product of X and Y . A diffeomorphism $f: (M, \nabla) \rightarrow (M, \bar{\nabla})$ is almost-geodesic if and only if the deformation tensor P satisfies

$$(5) \quad P(X_1, X_2, X_3) \wedge P(X_4, X_5) \wedge X_6 = 0 \text{ for all } X_i \in \mathcal{X}(M), \quad i = 1, \dots, 6.$$

In local coordinates, (5) reads $P_{(pqr)}^{[h} P_{su}^i \delta_v^j] = 0$ where the round and square brackets denote symmetrization and alternation of indices, respectively.

3. CLASSIFICATION OF ALMOST GEODESIC MAPPINGS

N.S. Sinyukov distinguished three kinds of almost geodesic mappings, [5, 6], namely π_1 , π_2 , and π_3 , characterized, respectively, by the conditions for the deformation tensor:

$$\pi_1: P(X, X, X) + P(P(X, X), X) = a(X, X) \cdot X + b(X) \cdot P(X, X), \quad X \in \mathcal{X}(M),$$

where a is a symmetric type $(0, 2)$ tensor field and b is a one-form;

$$\pi_2: P(X, X) = \psi(X) \cdot X + \varphi(X) \cdot F(X), \quad X \in \mathcal{X}(M),$$

where ψ and ϕ are one-forms, and F is a type $(1, 1)$ tensor field satisfying

$$(\nabla F)(X; X) + F(F(X), X) = \mu(X) \cdot X + \varrho(X) \cdot F(X), \quad X \in \mathcal{X}(M)$$

for some one-forms μ , ϱ ;

$$\pi_3: P(X, X) = \psi(X) \cdot X + a(X, X) \cdot Z, \quad X \in \mathcal{X}(M)$$

where ψ is a one-form, a is a symmetric bilinear form and $Z \in \mathcal{X}(M)$ is a vector field satisfying

$$(\nabla Z)(X) = h \cdot X + \theta(X) \cdot Z$$

for some scalar function $h: M \rightarrow \mathbb{R}$ and some one-form θ .

The so-called $\tilde{\pi}_1$ -mappings, *canonical* almost geodesic mappings, are characterized among almost geodesic mappings by the condition $b = 0$ on the right hand side. That is, the deformation tensor of a $\tilde{\pi}_1$ -mapping satisfies

$$(6) \quad P(X, X, X) + P(P(X, X), X) = a(X, X) \cdot X, \quad X \in \mathcal{X}(M).$$

It is known that any π_1 -mapping arises as a composition of a $\tilde{\pi}_1$ -mapping and a geodesic one. But geodesic mappings can be considered as trivial almost

geodesic mappings, and can be omitted in our further considerations; they have been analysed and classified in [1]. Our aim is to study $\tilde{\pi}_1$ -mappings of affine manifolds onto particular types of Riemannian spaces, namely those cases that induce integrable systems.

4. RICCI-SYMMETRIC AND GENERALIZED RICCI-SYMMETRIC MANIFOLDS

Under a *Ricci-symmetric manifold (space)* we mean a manifold (M, ∇) with linear connection (a pseudo-Riemannian space (M, g) , respectively) for which the Ricci tensor is parallel (=covariantly constant),

$$\nabla \text{Ric} = 0.$$

It was proven in [6] that the family of all $\tilde{\pi}_1$ -mappings of a manifold (M, ∇) (= "affine manifold") onto Ricci-symmetric (pseudo-)Riemannian spaces $(\bar{M}, \bar{g})$ ($\bar{\nabla} \text{Ric} = 0$) is given by an integrable system of differentiable equations (in covariant derivatives). Consequently, given a manifold with a symmetric connection, the family of all Ricci-symmetric Riemannian spaces $(\bar{M}, \bar{g})$ which can serve as images of the given manifold (M, ∇) under some $\tilde{\pi}_1$ -mapping, depends on a finite set of parameters.

On the other hand, geodesic mappings form a subset in the set of $\tilde{\pi}_1$ -mappings; they obey the definition. But basic equations describing geodesic mappings of a manifold with linear connection do not form an integrable system of Cauchy type, since a general solution depends on n arbitrary functions. It follows that the conditions (6) describing $\tilde{\pi}_1$ -mappings (i.e. canonical almost geodesic mappings) of affine manifolds do not, in general, induce an integrable system.

In the following, we consider a particular case when (6) can be transformed into an integrable system, generalizing the results of Sinyukov. Namely, we will investigate $\tilde{\pi}_1$ -mappings of an affine manifold (M, ∇) onto the so-called generalized Ricci-symmetric manifolds.

An affine manifold (M, ∇) will be called a *generalized Ricci-symmetric manifold* if its Ricci tensor satisfies

$$(7) \quad \nabla \text{Ric}(Y, Z; X) + \nabla \text{Ric}(X, Z; Y) = 0,$$

that is, $\nabla_X \text{Ric}(Y, Z) = -\nabla_Y \text{Ric}(X, Z)$. We do not a priori suppose the Ricci tensor be symmetric. If Ric is symmetric and (7) holds then Ric is parallel, $\nabla \text{Ric} = 0$, and (M, ∇) is a Ricci-symmetric manifold. Einstein spaces (Riemannian spaces characterized by the property that the Ricci tensor is proportional to the metric tensor) satisfy (7) since they satisfy $\nabla \text{Ric} = 0$, hence are generalized Ricci-symmetric. In this sense, generalized Ricci-symmetric spaces can be considered as a certain generalization of Einstein spaces.

5. ALMOST GEODESIC MAPPINGS $\tilde{\pi}_1$ ONTO GENERALIZED RICCI-SYMMETRIC MANIFOLDS

Given affine m -dimensional manifolds $\mathbb{A} = (M, \nabla)$ and $\bar{\mathbb{A}} = (\bar{M}, \bar{\nabla})$ with the corresponding curvature tensors R and $\bar{R}$, respectively, all connection-preserving mappings $f: M \rightarrow \bar{M}$ can be described by the following system of (differential) equations, [6, 7, 8]:

$$\begin{aligned} 3(\nabla_Z P(X, Y) + P(Z, P(X, Y))) \\ = \sum_{CS(X, Y)} (R(Y, Z)X - \bar{R}(Y, Z)X) + \sum_{CS(X, Y, Z)} a(X, Y)Z. \end{aligned}$$

It becomes clear that the above invariant formulas are rather complicated. As for the rest, we prefer to express our equalities in local coordinates (with respect to a map (U, φ) on M). This formulas have the following local expression

$$(8) \quad 3(P_{ij,k}^h + P_{k\alpha}^h P_{ij}^\alpha) = R_{(ij)k}^h - \bar{R}_{(ij)k}^h + a_{(ij)}\delta_k^h,$$

where P_{ij}^h , a_{ij} , R_{ijk}^h , $\bar{R}_{ijk}^h$ are local components of tensors P , R , $\bar{R}$ and a , respectively, δ_k^h is the Kronecker delta, “,” denotes covariant derivative with respect to ∇ .

The system (8) can be considered as a system of partial differential equations for functions P_{ij}^h on M , i.e. for components of the deformation tensor; the corresponding integrability conditions are

$$\begin{aligned} \bar{R}_{(ij)[k,\ell]}^h = R_{(ij)[k,\ell]}^h + \delta_{(i}^h a_{jk),\ell} - \delta_{(i}^h a_{j\ell),k} - 3(-P_{ij}^\alpha \bar{R}_{\alpha k\ell}^h + P_{\alpha(j}^h R_{i)k\ell}^\alpha) \\ - P_{\alpha k}^h (R_{(ij)\ell}^\alpha - \bar{R}_{(ij)\ell}^\alpha \delta_{(i}^\alpha a_{j\ell)}) + P_{\alpha\ell}^h (R_{(ij)k}^\alpha - \bar{R}_{(ij)k}^\alpha \delta_{(i}^\alpha a_{jk)}). \end{aligned}$$

Passing from $\nabla \bar{R}$ to $\bar{\nabla} \bar{R}$ on the left hand side we get the following integrability conditions of the system (8):

$$(9) \quad \bar{R}_{(ij)[k;\ell]}^h = \delta_{(i}^h a_{jk),\ell} - \delta_{(i}^h a_{j\ell),k} + \Theta_{ijk\ell}^h,$$

where

$$\begin{aligned} \Theta_{ijk\ell}^h = R_{(ij)[k,\ell]}^h - 3(-P_{ij}^\alpha \bar{R}_{\alpha k\ell}^h + P_{\alpha(j}^h R_{i)k\ell}^\alpha) \\ - P_{\alpha k}^h (R_{(ij)\ell}^\alpha - \bar{R}_{(ij)\ell}^\alpha \delta_{(i}^\alpha a_{j\ell)}) + P_{\alpha\ell}^h (R_{(ij)k}^\alpha - \bar{R}_{(ij)k}^\alpha \delta_{(i}^\alpha a_{jk)}) \\ - P_{\ell(i}^\alpha \bar{R}_{|\alpha|j)k}^h - P_{\ell(i}^\alpha \bar{R}_{j)\alpha k}^h + P_{k(i}^\alpha \bar{R}_{|\alpha|j)\ell}^h + P_{k(i}^\alpha \bar{R}_{j)\alpha\ell}^h. \end{aligned}$$

Using the Bianchi identity we can write (9) in local coordinate as

$$\bar{R}_{ilk;j}^h + \bar{R}_{j\ell k;i}^h = \delta_{(i}^h a_{jk),\ell} - \delta_{(i}^h a_{j\ell),k} + \Theta_{ijk\ell}^h,$$

where “;” denotes covariant derivative with respect to $\bar{\nabla}$. Contraction in h and k gives the following equality for covariant derivatives of components of the Ricci tensor $\bar{\text{Ric}}$ of $\bar{\nabla}$:

$$(10) \quad \bar{R}_{i\ell;j} + \bar{R}_{j\ell;i} = (n+1)a_{ij,\ell} - a_{\ell(i,j)} + \Theta_{ij\alpha\ell}^\alpha.$$

In the following let us suppose that the affine manifold $(\bar{M}, \bar{\nabla})$ is a generalized Ricci-symmetric space, that is, (7) holds. In local coordinates, (7) reads

$$\bar{R}_{ij;k} + \bar{R}_{kj;i} = 0.$$

Under this assumption, (10) reads

$$(11) \quad (n+1)a_{ij,\ell} - a_{\ell i,j} - a_{\ell j,i} = -\Theta_{ij\alpha}^\alpha.$$

Using symmetrization in ℓ, i gives

$$a_{\ell i,j} + a_{\ell j,i} = -\frac{1}{n}\Theta_{(i|\ell\alpha|j)}^\alpha + \frac{2}{n}a_{ij,\ell}.$$

Now (11) reads

$$(12) \quad \frac{n^2 + n - 2}{n} a_{ij,\ell} = -\Theta_{ij\alpha}^\alpha - \frac{1}{n}\Theta_{(i|\ell\alpha|j)}^\alpha.$$

Applying covariant differentiation with respect to $\bar{\nabla}$ to the integrability conditions (9), followed by passing from covariant derivative $\bar{\nabla}$ to ∇ on the right hand side, we get

$$(13) \quad \bar{R}_{(ij)k;\ell m}^h - \bar{R}_{(ij)\ell;mk}^h = \delta_{(i}^h a_{jk),\ell m} - \delta_{(i}^h a_{j\ell),km} + T_{ijk\ell m}^h,$$

where

$$\begin{aligned} T_{ijk\ell m}^h &= \bar{R}_{\alpha mk}^h \bar{R}_{(ij)\ell}^\alpha - \bar{R}_{\ell mk}^\alpha \bar{R}_{(ij)\alpha}^h - \bar{R}_{jmk}^\alpha \bar{R}_{(i\alpha)\ell}^h - \bar{R}_{imk}^\alpha \bar{R}_{(j\alpha)\ell}^h \\ &\quad - P_{m\alpha}^h \delta_{(i}^h a_{jk),\ell} - P_{mj}^\alpha \delta_{(i}^h a_{\alpha k),\ell} - P_{mi}^\alpha \delta_{(\alpha}^h a_{jk),\ell} - P_{mk}^\alpha \delta_{(\alpha}^h a_{ij),\ell} - P_{ml}^\alpha \delta_{(i}^h a_{jk),\alpha} \\ &\quad - P_{m\alpha}^h \delta_{(i}^h a_{j\ell),k} + P_{mi}^\alpha \delta_{(\alpha}^h a_{j\ell),k} + P_{mj}^\alpha \delta_{(i}^h a_{\alpha\ell),k} + P_{mk}^\alpha \delta_{(i}^h a_{j\ell),\alpha} - P_{ml}^\alpha \delta_{(i}^h a_{j\alpha),k} \\ &\quad - \theta_{ijk\ell,m}^h + P_{\alpha m}^h \theta_{ijk\ell}^\alpha - P_{mi}^\alpha \theta_{\alpha jk\ell}^h - P_{mj}^\alpha \theta_{i\alpha k\ell}^h - P_{mk}^\alpha \theta_{ij\alpha\ell}^h - P_{ml}^\alpha \theta_{ijk\alpha}^h. \end{aligned}$$

Alternating (13) in ℓ, m we obtain

$$\begin{aligned} (14) \quad &\bar{R}_{(ij)m;\ell k}^h - \bar{R}_{(ij)\ell;mk}^h = \delta_{(i}^h a_{jm),k\ell} - \delta_{(i}^h a_{j\ell),km} + T_{ijk[\ell m]}^h \\ &+ \bar{R}_{(i|\alpha k|}^h \bar{R}_{j)m\ell}^\alpha + \bar{R}_{(ij)\alpha}^h \bar{R}_{km\ell}^\alpha - \bar{R}_{(ij)k}^\alpha \bar{R}_{\alpha m\ell}^h + \bar{R}_{\alpha(i|k|}^h \bar{R}_{j)m\ell}^\alpha \\ &+ \delta_{(\alpha}^h a_{jk)R_{i\ell m}^\alpha} + \delta_{(\alpha}^h a_{ik)R_{j\ell m}^\alpha} + \delta_{(i}^h a_{j\alpha)R_{k\ell m}^\alpha} - \delta_{(i}^h a_{jk)R_{\alpha\ell m}^\alpha}. \end{aligned}$$

Due to the properties of the Riemannian tensor, (14) can be written as

$$(15) \quad \bar{R}_{im\ell;jk}^h + \bar{R}_{jm\ell;ik}^h = \delta_{(i}^h a_{j\ell),km} - \delta_{(i}^h a_{jm),k\ell} - N_{ijk\ell m}^h,$$

where

$$\begin{aligned} N_{ijk\ell m}^h &= T_{ijk[\ell m]}^h + \bar{R}_{im\ell}^\alpha \bar{R}_{(\alpha j)k}^h + \bar{R}_{jm\ell}^\alpha \bar{R}_{(\alpha i)k}^h + \bar{R}_{km\ell}^\alpha \bar{R}_{(ij)\alpha}^h \\ &\quad - \bar{R}_{\alpha m\ell}^h \bar{R}_{(ij)k}^\alpha + \delta_{(\alpha}^h a_{jk)R_{i\ell m}^\alpha} + \delta_{(\alpha}^h a_{ik)R_{j\ell m}^\alpha} + \delta_{(\alpha}^h a_{ij)R_{k\ell m}^\alpha} - a_{(ij)R_k)_{\ell m}^h. \end{aligned}$$

Let us alternate (15) in j, k . We get

$$\begin{aligned} (16) \quad &\bar{R}_{j m \ell ; i k}^h - \bar{R}_{k m \ell ; i j}^h = \delta_{(i}^h a_{j \ell), k m} - \delta_{(i}^h a_{j m), k \ell} - \delta_{(i}^h a_{k \ell), j m} + \delta_{(i}^h a_{k m), j \ell} \\ &- N_{i[j k] \ell m}^h + \bar{R}_{\alpha m \ell}^h \bar{R}_{i k j}^\alpha + \bar{R}_{i \alpha \ell}^h \bar{R}_{m k j}^\alpha + \bar{R}_{i m \alpha}^h \bar{R}_{\ell k j}^\alpha - \bar{R}_{i m \ell}^\alpha \bar{R}_{\alpha k j}^h. \end{aligned}$$

Let us interchange i and k in (15), and then use (16). We evaluate

$$(17) \quad \begin{aligned} 2\bar{R}_{jml;ik}^h &= \delta_{(i}^h a_{j\ell),km} - \delta_{(i}^h a_{jm),k\ell} - \delta_{(k}^h a_{jm),i\ell} \\ &\quad + \delta_{(i}^h a_{km),j\ell} - \delta_{(i}^h a_{k\ell),jm} + \delta_{(j\ell}^h a_{k),im} + \Omega_{ijk\ell m}^h, \end{aligned}$$

where

$$\begin{aligned} \Omega_{ijk\ell m}^h &= -N_{ijk\ell m}^h + N_{k[ij]k\ell m}^h - \bar{R}_{\alpha ml}^h \bar{R}_{(kj)i}^\alpha + \bar{R}_{j\alpha\ell}^h \bar{R}_{mik}^\alpha + \bar{R}_{jm\alpha}^h \bar{R}_{\ell ik}^\alpha \\ &\quad - \bar{R}_{\alpha i(j} \bar{R}_{k)m\ell}^\alpha + \bar{R}_{j\alpha\ell}^h \bar{R}_{mik}^\alpha + \bar{R}_{jm\alpha}^h \bar{R}_{\ell ik}^\alpha - \bar{R}_{\alpha ml}^h \bar{R}_{ikj}^\alpha - \bar{R}_{i\alpha\ell}^h \bar{R}_{mkj}^\alpha + \bar{R}_{im[\ell} \bar{R}_{\alpha]kj}^h. \end{aligned}$$

On the left hand side of (17), let us pass from covariant derivative with respect to $\bar{\nabla}$ to ∇ :

$$(18) \quad \begin{aligned} 2\bar{R}_{jml;ik}^h &= \delta_{(i}^h a_{j\ell),km} - \delta_{(i}^h a_{jm),k\ell} - \delta_{(k}^h a_{jm),i\ell} \\ &\quad + \delta_{(i}^h a_{km),j\ell} - \delta_{(i}^h a_{k\ell),jm} - \delta_{(k}^h a_{j\ell),im} + S_{ijk\ell m}^h, \end{aligned}$$

where

$$\begin{aligned} S_{ijk\ell m}^h &= \Omega_{ijk\ell m}^h - 2[\bar{R}_{jml,i}^\alpha P_{\ell k}^h - \bar{R}_{\alpha ml,i}^h P_{jk}^\alpha \\ &\quad - \bar{R}_{j\alpha\ell,i}^h P_{mk}^\alpha - \bar{R}_{jm\alpha,i}^h P_{\ell k}^\alpha - \bar{R}_{jml,\alpha}^h P_{ik}^\alpha \\ &\quad - (\bar{R}_{jml}^\alpha P_{\alpha i}^\beta - \bar{R}_{\alpha ml}^h P_{ij}^\alpha - \bar{R}_{j\alpha\ell}^h P_{im}^\alpha - \bar{R}_{jm\alpha}^h P_{i\ell}^\alpha) P_{\underline{k}}^h \\ &\quad - (\bar{R}_{jml}^\alpha P_{\alpha\beta}^h - \bar{R}_{\alpha ml}^h P_{\beta j}^\alpha - \bar{R}_{j\alpha\ell}^h P_{\beta m}^\alpha - \bar{R}_{jm\alpha}^h P_{\beta\ell}^\alpha) P_{ik}^\beta \\ &\quad - (\bar{R}_{\underline{m}\ell}^\alpha P_{\alpha i}^h - \bar{R}_{\alpha ml}^h P_{\underline{i}}^\alpha - \bar{R}_{\beta\alpha\ell}^h P_{im}^\alpha - \bar{R}_{\beta m\alpha}^h P_{i\ell}^\alpha) P_{jk}^\beta \\ &\quad - (\bar{R}_{j\beta\ell}^\alpha P_{\alpha i}^h - \bar{R}_{\alpha\beta\ell}^h P_{ji}^\alpha - \bar{R}_{j\alpha\ell}^h P_{\underline{i}}^\alpha - \bar{R}_{j\beta\alpha}^h P_{i\ell}^\alpha) P_{km}^\beta \\ &\quad - (\bar{R}_{jm\beta}^\alpha P_{\alpha i}^h - \bar{R}_{\alpha m\beta}^h P_{ji}^\alpha - \bar{R}_{j\alpha\beta}^h P_{mi}^\alpha - \bar{R}_{jm\alpha}^h P_{\underline{i}}^\alpha) P_{k\ell}^\beta]. \end{aligned}$$

Denoting $R_{jml i}^h = \bar{R}_{jml,i}^h$, i. e. introducing a new tensor field of type $(1, 4)$ we can write the system (18) in the following form

$$(19) \quad \bar{R}_{jml,i}^h = R_{jml i}^h$$

and

$$(20) \quad \begin{aligned} 2R_{jml i,k}^h &= \delta_{(i}^h a_{j\ell),km} - \delta_{(i}^h a_{jm),k\ell} - \delta_{(k}^h a_{jm),i\ell} \\ &\quad + \delta_{(i}^h a_{km),j\ell} - \delta_{(i}^h a_{k\ell),jm} + \delta_{(k}^h a_{j\ell),im} + S_{ijk\ell m}^h, \end{aligned}$$

where we used (12).

It can be verified that the equations (8), (12), (19) and (20) for functions $P_{ij}^h(x)$, $a_{ij}(x)$, $\bar{R}_{ijk}^h(x)$ and $R_{ijkm}^h(x)$ on (M, ∇) form an integrable system; the above functions must satisfy also additional algebraic conditions

$$(21) \quad \begin{aligned} P_{ij}^h(x) &= P_{ji}^h(x), \quad a_{ij}(x) = a_{ji}(x), \quad \bar{R}_{i(jk)}^h(x) = \bar{R}_{(ijk)}^h(x) = 0, \\ R_{i(jk)\ell}^h(x) &= R_{(ijk)\ell}^h(x) = 0. \end{aligned}$$

So we have succeeded to prove the following generalization of the result of Sinyukov [7, 8] (we use the above notation).

Theorem. *Let (M, ∇) be a manifold with affine connection and $(\bar{M}, \bar{\nabla})$ a generalized Ricci-symmetric manifold. There is a $\tilde{\pi}_1$ mapping $f: M \rightarrow \bar{M}$ (i.e. a canonical almost geodesic mapping of type π_1) if and only if there exist functions $P_{ij}^h(x)$, $a_{ij}(x)$, $\bar{R}_{ijk}^h(x)$ and $R_{ijkm}^h(x)$ which satisfy the equations (8), (12), (19), (20), and (21). The system of equations (8), (12), (19) and (20) forms a Cauchy type system of PDE's in covariant derivatives.*

As a consequence we obtain

Corollary. *The family of all generalized Ricci-symmetric manifolds, which can serve as an image of the given affine manifold (M, ∇) under some $\tilde{\pi}_1$ -mapping, depends on at most*

$$(22) \quad \frac{1}{6} n(n+1)(2n^3 - 4n^2 + 5n + 3)$$

parameters.

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