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ON A CLASS OF CURVATURE PRESERVING ALMOST GEODESIC MAPPINGS OF MANIFOLDS WITH AFFINE CONNECTION

BEREZOVSKI Vladimir, (UKR), MIKEŠ Josef, (CZ), VANŽUROVÁ Alena, (CZ)

Abstract. In this paper we pay attention to a particular case of almost geodesic mappings of the first type between (differentiable) manifolds with affine connection. We use here classical tensor methods and the apparatus of partial differential equations.
We prove that under the mappings under consideration, the invariant geometric object is just the (Riemannian) curvature tensor of the connection. We present the basic equations of the class of mappings under consideration in an equivalent form of the Cauchy system in covariant derivatives.

Key words and phrases. almost geodesic mappings, invariant geometric object, manifolds with affine connection.

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1 Introduction

Many monographs and papers are devoted to the theory of geodesic and holomorphically projective mappings, see [1]-[18]. We continue here a research project on geodesic and almost geodesic mappings of spaces with affine connection, or pseudo-Rimannian spaces, respectively.

In this paper we pay attention to a particular case of almost geodesic mappings of the first type between (differentiable) manifolds endowed affine connection. We prove that under such maps, the invariant geometric object is just the (Riemannian) curvature tensor of the connection. We present the basic equations of the class of mappings under consideration in an equivalent form of the Cauchy system in covariant derivatives.

As the main tool, we use here classical tensor methods and the apparatus of partial differential equations.

2 Almost geodesic mappings of manifolds with affine connection

Let us recall the basic concepts of the theory of almost geodesic mappings of manifolds with affine connection introduced in [12, 13, 14]. A (differentiable) curve defined in a manifold with affine connection A_n is called *almost geodesic* if there is a (differentiable) two-dimensional parallel distribution along the curve such that the tangent vectors of the curve, being parallely transported along the curve, still belong to the distribution.

A diffeomorphism $f\colon A_n \to \bar{A}_n$ of manifolds with affine connection is called *almost geodesic* if all geodesics in A_n are mapped onto almost geodesic curves of $\bar{A}_n$.

A map of A_n onto $\bar{A}_n$ is almost geodesic if and only if in a common coordinate system $(x^1, \ldots, x^n)$ (with respect to the diffeomorphism f, [11, p. 85]), the deformation tensor of the connections ([11, p. 86]) $P^h_{ij}(x) = \bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x)$ satisfies

$$A^h_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma \equiv aP^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + b\lambda^h$$

where $A^h_{ijk} \equiv P^h_{ij,k} + P^\alpha_{ij}P^h_{\alpha k}$, Γ^h_{ij} ($\bar{\Gamma}^h_{ij}$, respectively) are components of the connection in the manifold A_n ($\bar{A}_n$, respectively), λ^h is an arbitrary vector, and a, b are some functions of the variables x^h, λ^h. Here and in what follows, "," denotes the covariant derivative with respect to the connection of A_n.

N.S. Sinyukov distinguished in [8, 12, 13, 14] three types of almost geodesic mappings denoted by π_1, π_2 and π_3. We proved in [1, 8] that for dimensions $n > 5$, there are no others. Almost geodesic mappings (in short, AGM) of type π_1 are characterized by the following conditions for the deformation tensor

$$A^h_{(ijk)} = \delta^h_{(i}a_{jk)} + b_{(i}P^h_{jk)}$$

where a_{ij} is a symmetric tensor, b_i is a covector, δ^h_i is the Kronecker tensor, and (ijk) means symmetrization (without division) with respect to the listed indices.

3 A particular subclass of the first type AGM

Let then following conditions are satisfied under the diffeomorphism of manifolds with affine connection:

$$P^h_{ij,k} = -P^\alpha_{ij}P^h_{\alpha k} + \delta^h_{(k}a_{ij)} \tag{1}$$

Such mappings belong, as a particular case, to AGM of the first type. The Riemannian tensors R^h_{ijk} and $\bar{R}^h_{ijk}$ of the manifolds A_n and $\bar{A}_n$, respectively, are related by [13]

$$\bar{R}^h_{ijk} = R^h_{ijk} + P^h_{i[k,j]} + P^\alpha_{i[k}P^h_{j]\alpha} \tag{2}$$

where $[kj]$ denotes alternation with respect to the distinguished indices. According to (1) and (2), the following holds:

Theorem 3.1 *The Riemannian tensor R^h_{ijk} is an invariant object of manifolds with affine connection under almost geodesic mappings satisfying (1).*

Since the Riemmanian tensor vanishes in affine spaces, we deduce that the affine spaces form a class closed under AGM satisfying (1):

Theorem 3.2 *If an affine space A_n admits an almost geodesic mapping onto $\bar{A}_n$ that satisfies (1), then $\bar{A}_n$ is an affine space.*

Regarding (1) as a system of Cauchy type with respect to the components of the deformation tensor P^h_{ij} we find the corresponding integrability conditions. For this purpose, let us calculate covariant derivatives of (1) with respect to x^m, and let us alternate in k and m. Accounting the Ricci identity we get

$$\delta^h_i a_{j[k,m]} + \delta^h_j a_{i[k,m]} + \delta^h_{[k|} a_{ij,|m])} = -P^\alpha_{ij} R^h_{\alpha km} + P^h_{\alpha(j} R^\alpha_{i)km} + a_{j[m} P^h_{k]i} + a_{i[m} P^h_{k]j}. \tag{3}$$

Now contracting the integrability conditions in h and m we get

$$a_{jk,i} + a_{ik,j} - (n+1) a_{ij,k} = -P^\alpha_{ij} R_{\alpha k} + P^\beta_{\alpha j} R^\alpha_{ik\beta} + P^\beta_{\alpha i} R^\alpha_{jk\beta} + a_{j\alpha} P^\alpha_{ki} - a_{jk} P^\alpha_{\alpha i} + a_{i\alpha} P^\alpha_{jk} - a_{ik} P^\alpha_{j\alpha}. \tag{4}$$

Further, alternating (4) over k and j we obtain

$$\begin{aligned} a_{ij,k} = \;& a_{ik,j} + \frac{1}{n+2}(P^\alpha_{ij} R_{\alpha k} - P^\alpha_{ik} R_{\alpha j} - P^\beta_{\alpha j} R^\alpha_{ik\beta} - P^\beta_{\alpha i} R^\alpha_{jk\beta} \\ & + P^\beta_{\alpha k} R^\alpha_{ij\beta} + P^\beta_{\alpha i} R^\alpha_{kj\beta} - a_{j\alpha} P^\alpha_{ki} + a_{k\alpha} P^\alpha_{ij} + a_{ik} P^\alpha_{j\alpha} - a_{ij} P^\alpha_{k\alpha}). \end{aligned} \tag{5}$$

In (5), let us interchange the indices k and i,

$$\begin{aligned} a_{kj,i} = \;& a_{ik,j} + \frac{1}{n+2}(P^\alpha_{kj} R_{\alpha i} - P^\alpha_{ki} R_{\alpha j} - P^\beta_{\alpha j} R^\alpha_{ki\beta} - P^\beta_{\alpha k} R^\alpha_{ji\beta} + P^\beta_{\alpha i} R^\alpha_{kj\beta} + P^\beta_{\alpha k} R^\alpha_{ij\beta} \\ & - a_{j\alpha} P^\alpha_{ik} + a_{i\alpha} P^\alpha_{kj} + a_{ki} P^\alpha_{j\alpha} - a_{kj} P^\alpha_{i\alpha}). \end{aligned} \tag{6}$$

Plugging (5) and (6) to (4) we find

$$\begin{aligned} a_{ik,j} = \;& \frac{1}{(n-1)(n+2)}[n(P^\alpha_{ik} R_{\alpha j} - P^\beta_{\alpha(k} R^\alpha_{i)j\beta}) + R_{\alpha(k} P^\alpha_{i)j} - P^\beta_{\alpha j} R^\alpha_{(ik)\beta} - P^\beta_{\alpha(i} R^\alpha_{|j|k)\beta} \\ & + (n+1)(a_{j(i} P^\alpha_{k)\alpha} - a_{\alpha(i} P^\alpha_{k)j} + 2(a_{ik} P^\alpha_{j\alpha} - a_{j\alpha} P^\alpha_{ik})]. \end{aligned} \tag{7}$$

Obviously, the equations (1) and (7) in the given space A_n respresent a Cauchy system in the functions $P^h_{ij}(x)$ and $a_{ij}(x)$ which, naturally, must satisfy also the following system of conditions of an algebraic character

$$P^h_{ij}(x) = P^h_{ji}(x), \qquad a_{ij}(x) = a_{ji}(x). \tag{8}$$

Hence we have proved:

Theorem 3.3 *A manifold with affine connection A_n admits almost geodesic mappings, satisfying the equation (1), onto a manifold with affine connection $\bar{A}_n$ if and only if there exists, in A_n, a solution of the mixed system of Cauchy type (1), (7) and (8) in the functions P^h_{ij} and a_{ij}.*

It is proved that the number of relevant parameters on which the solution of a system under consideration depends has the upper boundary

$$r \leq \frac{1}{2} n(n+1)^2.$$

4 An example of a particular subclass of the first type AGM

It the tensor a_{ij} vanishes identically the equation (1) reads

$$P^h_{ij,k} = -P^\alpha_{ij} P^h_{\alpha k}. \tag{9}$$

The equations (9) are completely integrable in a manifolds with affine connection. That is, the system is solvable for any initial conditions $P^h_{ij}(x_0)$. If we choose initial values satisfying $P^h_{ij}(X_0) \neq \delta^h_{(i}\psi_{j)}(x_0)$ then the obtained solution determines an almost geodesic map of the first type of an affine space A_n onto an affine space $\bar{A}_n$ that is not a geodesic one. Hence we have as a consequence

Theorem 4.1 *There exists an almost geodesic map of the first type of the affine space onto itself under which all straight lines are mapped onto plane curves not all of which are straight lines.*

Let $(x^1, \ldots, x^n)$ and $(\bar{x}^1, \ldots, \bar{x}^n)$ be affine coordinates in affine spaces A_n and $\bar{A}_n$, respectively. We give here a particular example of an almost geodesic map of the first type of a flat space A_n onto a flat space $\bar{A}_n$ as follows. Pointwise, the map is given in coordinates by

$$\bar{x}^h = \frac{1}{2}\, C^h_\alpha\, (x^\alpha - C^\alpha)^2 + x^h_0, \tag{10}$$

where C^h_i, C^h, x^h_0 are constants such that $x^h \neq C^h$, $\det(C^h_i) \neq 0$. It can be checked directly that the only non-zero components of the deformation tensor are

$$P^i_{ii} = \frac{1}{x^i - C^i}, \qquad i = 1, \ldots, n.$$

It can be verified that the tensor P^h_{ij} with such components satisfies (9). At the same time, we realize that the map just constructed belongs neither to the type π_2 nor π_3. Under this mapping, straight lines of the space A_n, given by parametrizations $x^h = a^h + b^h t$ where t is a parameter, are mapped onto parabolas in $\bar{A}_n$ given by the equations

$$\bar{x}^h = D^h + E^h t + F^h t^2$$

where $D^h = \frac{1}{2}\, C^h_\alpha (a^\alpha - C^\alpha)^2$, $E^h = \frac{1}{2}\, C^h_\alpha (a^\alpha - C^\alpha) b^\alpha$, and $F^h = \frac{1}{2}\, C^h_\alpha (b^\alpha)^2$. The only exceptions come for those straight lines for which the vectors E^h and F^h happen to be collinear: if this is the case the image of such a line is a straight line again.

Finally let us note that the equations (10) generate a system of almost geodesic maps of type π_1 of a flat space if we consider the coefficients C^h_i, C^h and x^h_0 to be continuous parameters.

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References

[1] BEREZOVSKIJ, V.; MIKEŠ, J.: *On the classification of almost geodesic mappings of affine-connected spaces*. Differential geometry and its applications, Proc. Conf., Dubrovnik/Yugosl. 1988, pp. 41-48, 1989.

[2] BEREZOVSKIJ, V.; MIKEŠ, J.: *On special almost geodesic mappings of type π_1 of spaces with affine connection*. Acta Univ. Palacki. Olomuc., Fac. Rerum Nat., Math. 43, pp. 21-26, 2004.

[3] BEREZOVSKIJ, V.; MIKEŠ, J.; VANŽUROVÁ, A.: *Canonical almost geodesic mappings of type π_1 onto pseudo-Riemannian manifolds*. Diff. geometry and its appl. Proc. of the 10th Int. Conf. on differential geometry and its applications, DGA 2007, Olomouc, Czech Republic, August 27–31, 2007. Hackensack, NJ: World Scientific. 65-75 (2008).

[4] FORMELLA, S.: *A note on geodesic and almost geodesic mappings of homogeneous Riemannian manifolds*. Opusc. Math. 25, No. 2, 181-187 (2005).

[5] GRIGOR'EVA, T.I.: *Invariant geometric objects of the canonical the almost geodesic mapping $\pi_2(e = 0)$*. (Ukrainian. English summary) Ukr. Mat. Zh. 54, No. 10, 1329-1335 (2002); translation in Ukr. Math. J. 54, No. 10, 1602-1610 (2002).

[6] GRIGOR'EVA, T.I.: *Properties of parabolic Kählerian spaces admitting an almost geodesic mapping of the type π_2 with degenerate affinor structure*. Ukr. Mat. Zh. 59, No. 11, 1574-1579 (2007); translation in Ukr. Math. J. 59, No. 11, 1768-1775 (2007).

[7] HINTERLEITNER, I.; MIKEŠ, J.; STRÁNSKÁ, J.: *Infinitesimal F-planar transformations*. Russ. Math. 52, No. 4, 13-18 (2008); translation from Izv. Vyssh. Uchebn. Zaved., Mat. 2008, No. 4, 16–22 (2008).

[8] MIKEŠ, J.: *Holomorphically projective mappings and their generalizations*. J. Math. Sci., New York 89, No. 3, 1334-1353 (1998); translation from Itogi Nauki Tekh., Ser. Sovrem Mat. Prilozh., Temat. Obz. 30, 258-291 (1996).

[9] MIKEŠ, J.; BEREZOVSKIJ, V.: *Geodesic mappings of affine-connected spaces onto Riemannian spaces*. Diff. geometry and its appl. Proc. of a colloquium, held in Eger, Hungary, August 20-25, 1989, organized by the Jnos Bolyai Mathematical Society. Amsterdam: North-Holland Publishing Company. Colloq. Math. Soc. Jnos Bolyai. 56, 491-494 (1992).

[10] MIKEŠ, J.; POKORNÁ, O.; STARKO, G.: *On almost geodesic mappings $\pi_2(e)$ onto Riemannian spaces*. The proc. of the 23th winter school "Geometry and physics", Srní, Czech Republic, January 18–25, 2003. Palermo: Circolo Matematico di Palermo. Suppl. Rend. Circ. Mat. Palermo, II. Ser. 72, 151-157 (2004).

[11] MIKEŠ, J.; VANŽUROVÁ, A.; HINTERLEITNER, I.: *Geodesic mappings and some generalizations*. Olomouc: Palacký University, Faculty of Science. 304 p. (2009).

[12] SINYUKOV, N.S.: *Almost geodesic mappings of affinely connected and Riemannian spaces*. Sov. Math., Dokl. 4, 1086-1088 (1963); translation from Dokl. Akad. Nauk SSSR 151, 781-782 (1963).

[13] SINYUKOV, N.S.: *Geodesic mappings of Riemannian spaces*. (Russian) Moskva: Nauka. 256 p. R. 1.60 (1979).

[14] SINYUKOV, N.S.: *Almost-geodesic mappings of affinely connected and Riemann spaces*. J. Sov. Math. 25, 1235-1249 (1984); translation from Itogi Nauki Tekh., Ser. Probl. Geom. 13, 3-26 (1982).

[15] SOBCHUK, V.S.: *Interior almost geodesic mappings*. Sov. Math. 33, No.5, 97-101 (1989); translation from Izv. Vyssh. Uchebn. Zaved., Mat. 1989, No.5(324), 62-64 (1989).

[16] SOBCHUK, V.S.: *Intrinsic almost geodesic mapping.* Izv. Vyssh. Uchebn. Zaved., Mat. 1989, No.5(324), 62-64 (1989).

[17] SOBCHUK, V.S.; MIKEŠ, J.; POKORNÁ, O.: *On almost geodesic mappings π_2 between semisymmetric Riemannian spaces.* Novi Sad J. Math. 29, No.3, 309-312 (1999).

[18] VAVŘÍKOVÁ, H.; MIKEŠ, J.; POKORNÁ, O.; STARKO, G. *On fundamental equations of almost geodesic mappings of type $\pi_2(e)$*. Russ. Math. 51, No. 1, 8-12 (2007); translation from Izv. Vyssh. Uchebn. Zaved., Mat. 2007, No.1, 10-15 (2007).

Current addresses

Vladimir Berezovskij, Doc. CSc.
Dept. of Mathematics, University of Uman
Institutskaya 1, Uman, Ukraine
berez.volod@rambler.ru

Josef Mikeš, Prof. RNDr. DrSc.
Dept. Algebra and Geometry, Faculty of Science, Palacký University
17. listopadu 12, 77900 Olomouc, Czech Republic,
tel. +420-585 634 656 and mikes@inf.upol.cz.

Alena Vanžurová, Doc. RNDr., CSc.
Dept. of Algebra and Geometry, Faculty of Science, Palacký University
17. listopadu 12, 779 00 Olomouc, Czech Rep.,
Dept. of Math., Faculty of Civil Engineering, Brno University of Technology
Veveří 331/95, 602 00 Brno, Czech Republic,
tel. +420-585 634 759, vanzurov@inf.upol.cz, vanzurova.a@fce.vutbr.cz