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Development of the fuzzy sets theory: weak operations and extension principles

The paper considers the problems that arise when using the theory of fuzzy sets to solve applied problems. Unlike stochastic methods, which are based on statistical data, fuzzy set theory methods make sense to apply when statistical data are not available. In these cases, algorithms should be based on membership functions formed by experts who are specialists in this field of knowledge. Ideally, complete information about membership functions is required, but this is an impractical procedure. More often than not, even the most experienced expert can determine only their carriers or separate sets of the α-cuts for unknown fuzzy parameters of the system. Building complete membership functions of unknown fuzzy parameters on this basis is risky and unreliable. Therefore, the paper proposes an extension of the fuzzy sets theory axiomatics in order to introduce non-traditional (less demanding on the completeness of data on membership functions) extension principles and operations on fuzzy sets. The so-called α-weak operations on fuzzy sets are proposed, which are based on the use of separate sets of the α-cuts. It is also shown that all classical theorems of Cantor sets theory apply in the extended axiomatic theory. New extension principles of generalization have been introduced, which allow solving problems in conditions of significant uncertainty of information.

Keywords: Cantor set, fuzzy set, function of belonging, set of α-cut, core of fuzzy set, α-weak operation.

Introduction

It is well known that the concept of a fuzzy set, proposed by L. Zadeh in 1965 [1], immediately arouse great interest among mathematicians and scientists of other fields and stimulated the appearance of a large number of works in this direction. Just two years later, Gauguin extended this concept to L-fuzzy sets, and further introduced the interval fuzzy line, regular fuzzy numbers and fuzzy metric spaces, fuzzy topological spaces, fuzzy relations and mappings, concepts and theorems of fuzzy algebra [2–11]. All these works with slight variations are based on the well-known maximin extension principle (MMPG) Zadeh [1], which fully satisfied the researchers. The mathematical apparatus of fuzzy set theory (FST) began to be widely used both in physics [12,13] and in applied disciplines [14–18]. At the same time, there are quite a few applied problems for which the use of the maximin extension principle prevents their solution. The fact is that the application of MMPG requires complete information about the membership functions of fuzzy defined parameters of the task, and this, unfortunately, is often the almost impossible procedure. In these cases, even the most experienced expert can determine only their cores or α-cuts for the unknown fuzzy parameters of the system. Building complete membership functions of unknown fuzzy parameters on this basis is risky and unreliable.

Thus, it seems appropriate to expand the axiomatics of the fuzzy sets theory in order to introduce non-traditional (less demanding on the completeness of data on membership functions) extension principles and operations on fuzzy sets. In works [19, 20], an unconventional class of so-called α-weak operations on fuzzy sets was proposed for the first time, further, introducing new concepts, we will follow these works.

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Statement of the problem

All problems with uncertain parameters, which should be solved using fuzzy set theory methods, can be divided into two classes:

1. Problems with non-numerical input parameters.

In these problems, each of the non-numerical parameters corresponds to a certain logical variable (term), to which the expert assigns a membership function (performs fuzzification), then certain procedures are carried out with the assigned membership functions, and the defuzzification procedure is applied to the new membership functions obtained as a result. As a rule, the quality of these calculations significantly depends on the knowledge of experts in the subject of research and on the adequacy of fuzzification and defuzzification procedures.

2. Problems with non-numerical input parameters.

As a rule, it is advisable to solve such problems using the methods of probability theory, but for this the researcher must have a sufficient amount of reliable statistical data. If these data are not available, or their number is very small, then it makes sense to apply the methods of fuzzy set theory. In this case, the uncertain parameters are given by vague numbers, the membership functions of which are formed by experts who are specialists in this field of knowledge.

The main problem of these methods is that even the most experienced expert can determine only their cores or α-cuts for unknown fuzzy parameters of the system. Building complete membership functions of unknown fuzzy parameters on this basis is risky and unreliable.

Therefore, the task of expanding the axiomatics of the fuzzy sets theory in order to introduce non-traditional (less demanding on the completeness of data on membership functions) extension principles and operations on fuzzy sets is actual. For this, the authors propose to introduce the so-called α-weak operations on fuzzy sets, which are based on the use of α-cuts.

Research results

Let's consider the basics of weak operations axiomatics. The α-cut set of the fuzzy set $\tilde{A}$ defined on the universum X is the usual Cantor set of elements $x \in X$, for which the condition $\mu_{\tilde{A}}(x) \geq \alpha$ is fulfilled, where $\alpha \geq (0,1]$. The limiting case of the α-cut set is the so-called core (or, otherwise, the 0-cut) of the fuzzy set $\tilde{A}$, which is also a Cantor set of elements $x \in X$ for which the condition $\mu_{\tilde{A}}(x) > 0$ is fulfilled.

It is known that every operation on classical Cantor sets can be matched with many similar operations on fuzzy sets. There is only one mandatory condition that each of these operations must meet - they must reduce to the corresponding classical operation in the case of degeneracy of fuzzy sets to classical Cantor sets.

Obviously, that weak operations on fuzzy sets must have the same properties as the analogical ones on classical Cantor sets, that is the same theorems must be fair for them as for classical sets. Let's consider it on the example of the relation of loose inclusion. L. Zadeh defined this relation as: fuzzy set $\tilde{A}$, which is defined on the universum X, if and only if includes fuzzy set $\tilde{B}$, defined on this universum, when for all elements $x \in X$ the membership function $\mu_{\tilde{A}}(x)$ is more or equal to the membership function $\mu_{\tilde{B}}$

$$\tilde{A} \supseteq \tilde{B} \Leftrightarrow \forall x \geq X \left(\mu_{\tilde{A}}(x) \geq \mu_{\tilde{B}}(x)\right). \tag{1}$$

From the fuzzy theory point of view, the membership function of the classical Cantor set A in X looks like $\mu_A : X \to \{0,1\}$, and for the set A we can write

$$A = \{(x, \mu_A(x)) \,|\, \forall x \in X (x \in A \Leftrightarrow \mu_A(x) = 1)\}.$$

The definition of relation of inclusion for classical sets A and B, expressed through their membership function is formulated as: classical set A, defined on the universum X, if and only if includes classical set B, defined at the same universum, when for all elements $x \in X$, if $\mu_B(x) = 1$, then and $\mu_A(x) = 1$, that is

$$\tilde{A} \supseteq \tilde{B} \Leftrightarrow \forall x \in X(\mu_B(x) = 1 \Rightarrow \mu_A(x) = 1). \tag{2}$$

The definition, which lessens the demands to the membership functions $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$ in comparison with (1), doesn't demand the condition $\mu_{\tilde{A}}(x) \geq \mu_{\tilde{B}}(x)$ to be carried out, and is based on the sets of α-cuts of fuzzy set (which are the commom Cantor sets) and is suggested being called loose α-weak inclusion (is marked $\overset{\alpha}{\supseteq}$) and analogically can be formulated as (2): fuzzy set $\tilde{A}$, that defined on the universum X, α-weakly includes fuzzy set $\tilde{B}$, defined on the same universum, if and only if when for all elements $x \in X$, if $\mu_{\tilde{B}}(x) \geq \alpha$, then and $\mu_{\tilde{A}}(x) \geq \alpha$, or

$$\tilde{A} \overset{\alpha}{\supseteq} \tilde{B} \Leftrightarrow \forall x \in X \left(\mu_{\tilde{B}}(x) \geq \alpha \Rightarrow \mu_{\tilde{A}}(x) \geq \alpha\right).$$

In boundary case, the relation which is based on the cores of fuzzy sets $\tilde{A}, \tilde{B}$ is offered to call just loose weak inclusion or loose 0-weak inclusion (is marked $\overset{0}{\supseteq}$). Its definition can be formulated as: fuzzy set $\tilde{A}$, defined on the universum X, if and only if 0-weakly includes fuzzy set $\tilde{B}$, defined on the same universum, when for all elements $x \in X$, if $\mu_{\tilde{B}} > 0$, then and $\mu_{\tilde{A}} > 0$, or

$$\tilde{A} \overset{0}{\supseteq} \tilde{B} \Leftrightarrow \forall x \in X \left(\mu_{\tilde{B}}(x) > 0 \Rightarrow \mu_{\tilde{A}}(x) > 0\right).$$

Let's introduce the definition of the α-weak supplement operation. The traditional supplement of the fuzzy set $\tilde{A}$ in X is the accepted fuzzy set $\bar{\tilde{A}}$ is X, for which the following condition is carried out

$$\forall x \in X \left(\mu_{\bar{\tilde{A}}}(x) = 1 - \mu_{\tilde{A}}(x)\right).$$

For classical Cantor sets, the supplement of set A is considered to be the set $\overline{A}$, that is

$$\forall x \in X \left(\mu_A(x) = 1 \Leftrightarrow \mu_{\overline{A}}(x) = 0\right). \tag{3}$$

Analogically to (3) the definition of operation of α-weak supplement is offered to formulate as: fuzzy set $\overset{\alpha}{\bar{\tilde{A}}}$ in X is α-weak supplement of fuzzy set $\tilde{A}$ in X if and only if, when for all elements $x \in X$, if $\mu_{\tilde{A}}(x) \geq \alpha$, then $\mu_{\overset{\alpha}{\bar{\tilde{A}}}}(x) < \alpha$, and vice versa, that is

$$\forall x \in X \left(\mu_{\tilde{A}}(x) \geq \alpha \Leftrightarrow \mu_{\overset{\alpha}{\bar{\tilde{A}}}}(x) < \alpha\right). \tag{4}$$

It follows from (4) that

$$\forall x \in X \left(\mu_{\tilde{A}}(x) < \alpha \Leftrightarrow \mu_{\overset{\alpha}{\bar{\tilde{A}}}}(x) \geq \alpha\right).$$

Analogically to the definition (4) for the operation of weak supplement (or 0-weak supplement) we can write: fuzzy set $\overset{0}{\bar{\tilde{A}}}$ in X is a weak supplement of fuzzy set $\tilde{A}$ in X if and only if, when for all the elements $x \in X$, if $\mu_{\tilde{A}}(x) > 0$, then $\mu_{\bar{\tilde{A}}^0}(x) = 0$, and vice versa, that is

$$\forall x \in X \left(\mu_{\tilde{A}}(x) > 0 \Leftrightarrow \mu_{\bar{\tilde{A}}^0}(x) = 0 \right). \tag{5}$$

It follows from (5) that

$$\forall x \in X \left(\mu_{\tilde{A}}(x) = 0 \Leftrightarrow \mu_{\bar{\tilde{A}}^0}(x) > 0 \right).$$

The definition for the relation of α-weak equation between fuzzy sets $\tilde{A}, \tilde{B}$ in X is formulated as: fuzzy set $\tilde{A}$, defined on the universum X, α-weakly equal to fuzzy set $\tilde{B}$, defined on this universum, if and only if, when for all the elements $x \in X$, if $\mu_{\tilde{A}} > 0$, then and $\mu_{\bar{\tilde{A}}^\alpha}(x) = 0$, and vice versa, that is

$$\tilde{A} \overset{\alpha}{=} \tilde{B} \Leftrightarrow \forall x \in X \left(\mu_{\tilde{B}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \right).$$

For a weak equation (0-weak equation) we can write

$$\tilde{A} \overset{0}{=} \tilde{B} \Leftrightarrow \forall x \in X \left(\mu_{\tilde{B}}(x) > 0 \Leftrightarrow \mu_{\tilde{A}}(x) > 0 \right).$$

Let's consider the definition for other main relations between fuzzy sets and operations on them. It is suggested that α-weak combination of fuzzy sets $\tilde{A}$ and $\tilde{B}$ in X is the fuzzy set $\tilde{C} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cup} \tilde{B}$ in X, if and only if, when for all elements $x \in X$, if $\mu_{\tilde{C}}(x) \geq \alpha$ then $\mu_{\tilde{A}}(x) \geq \alpha$ or $\mu_{\tilde{B}}(x) \geq \alpha$, and vice versa, that is

$$\tilde{C} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cup} \tilde{B} \Leftrightarrow \forall x \in X \left(\mu_{\tilde{C}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \vee \mu_{\tilde{B}}(x) \geq \alpha \right).$$

Analogically, weak (0-weak) association of fuzzy sets $\tilde{A}$ and $\tilde{B}$ in X is the fuzzy set $\tilde{C} \overset{0}{=} \tilde{A} \overset{0}{\cup} \tilde{B}$ in X if and only if, when for all elements $x \in X$, if $\mu_{\tilde{C}}(x) > 0$ then $\mu_{\tilde{A}}(x) > 0$ or $\mu_{\tilde{B}}(x) > 0$, and vice versa, that is

$$\tilde{C} \overset{0}{=} \tilde{A} \overset{0}{\cup} \tilde{B} \Leftrightarrow \forall x \in X \left(\mu_{\tilde{C}}(x) > 0 \Leftrightarrow \mu_{\tilde{A}}(x) > 0 \vee \mu_{\tilde{B}}(x) > 0 \right).$$

At last, α-weak crossing of fuzzy sets $\tilde{A}$ and $\tilde{B}$ in X is the fuzzy set $\tilde{C} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cap} \tilde{B}$ in X if and only if, when for all elements $x \in X$, if $\mu_{\tilde{C}}(x) \geq \alpha$, then $\mu_{\tilde{A}}(x) \geq \alpha$ and $\mu_{\tilde{B}}(x) \geq \alpha$, and vice versa, that is

$$\tilde{C} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cap} \tilde{B} \Leftrightarrow \forall x \in X \left(\mu_{\tilde{C}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_{\tilde{B}}(x) \geq \alpha \right).$$

Analogically, a weak (0-weak) crossing of fuzzy sets $\tilde{A}$ and $\tilde{B}$ in X is the fuzzy set $\tilde{C} \overset{0}{=} \tilde{A} \overset{0}{\cap} \tilde{B}$ in X if and only if $\mu_{\tilde{C}}(x) > 0$, when for all elements $x \in X$, if then $\mu_{\tilde{A}}(x) > 0$ and $\mu_{\tilde{B}}(x) > 0$, and vice versa, that is

$$\tilde{C} \overset{0}{=} \tilde{A} \overset{0}{\cap} \tilde{B} \Leftrightarrow \forall x \in X \left(\mu_{\tilde{C}}(x) > 0 \Leftrightarrow \mu_{\tilde{A}}(x) > 0 \wedge \mu_{\tilde{B}}(x) > 0 \right).$$

The definition of the more complex operation of the Descartes multiplication of fuzzy sets is suggested as follows: α-weak Descartes multiplication of the fuzzy sets $\tilde{A}_i$ in X is the fuzzy set $\tilde{A} \overset{\alpha}{=} \tilde{A}_1 \overset{\alpha}{\times} \tilde{A}_2 \overset{\alpha}{\times} \ldots \cdots \overset{\alpha}{\times} \tilde{A}_n$ in $X = X_1 \times X_2 \times \ldots \ldots X_n$ if and only if, when for all elements $x = (x_1, x_2, \ldots, x_n) \in X$, if $\mu_{\tilde{A}}(x) \geq \alpha$, then simultaneously $\mu_{\tilde{A}_1}(x) \geq \alpha$, $\mu_{\tilde{A}_2}(x) \geq \alpha$, $\ldots, \mu_{\tilde{A}_n}(x) \geq \alpha$ and vice versa, that is

$$\tilde{A} \overset{\alpha}{=} \tilde{A}_1 \overset{\alpha}{\times} \tilde{A}_2 \overset{\alpha}{\times} \ldots \cdots \overset{\alpha}{\times} \tilde{A}_n \Leftrightarrow$$

$$\Leftrightarrow x = (x_1, x_2, \ldots, x_n) \in X \left(\mu_{\tilde{A}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}_1}(x) \geq \alpha \wedge \mu_{\tilde{A}_2}(x) \geq \alpha \wedge \cdots \wedge \mu_{\tilde{A}_n}(x) \geq \alpha\right).$$

Accordingly, weak (0-weak) Descartes multiplication of fuzzy sets $\tilde{A}_i$ in X is the fuzzy set $\tilde{A} \overset{0}{=} \tilde{A}_1 \overset{0}{\times} \tilde{A}_2 \overset{0}{\times} \ldots \cdots \overset{0}{\times} \tilde{A}_n$ in $X = X_1 \times X_2 \times \ldots \ldots X_n$ if and only if, when for all elements $x = (x_1, x_2, \ldots, x_n) \in X$, if $\mu_{\tilde{A}}(x) > 0$, then simultaneously $\mu_{\tilde{A}_1}(x) > 0$, $\mu_{\tilde{A}_2}(x) > 0$, $\ldots, \mu_{\tilde{A}_n}(x) > 0$, and vice versa, that is

$$\tilde{A} \overset{0}{=} \tilde{A}_1 \overset{0}{\times} \tilde{A}_2 \overset{0}{\times} \ldots \cdots \overset{0}{\times} \tilde{A}_n \Leftrightarrow$$

$$\Leftrightarrow x = (x_1, x_2, \ldots, x_n) \in X \left(\mu_{\tilde{A}}(x) > 0 \Leftrightarrow \mu_{\tilde{A}_1}(x) > 0 \wedge \mu_{\tilde{A}_2}(x) > 0 \wedge \cdots \wedge \mu_{\tilde{A}_n}(x) > 0\right).$$

If we analyze all the above definitions of α-weak operations, we can come to the conclusion that the results of α-weak operations are ambiguous. Unlike traditional operations on fuzzy sets, the result of any α-weak operation is not a specific fuzzy set, but a set of fuzzy sets, each of which satisfies given conditions. This ambiguity makes it possible to operate with fuzzy sets, the membership functions of which are not completely specified or are specified imprecisely. Such functions are most often obtained with the help of expert procedures.

It is obvious that α-weak operations on fuzzy sets should have the same properties as similar operations on classical Cantor sets, that is, the same theorems as for classical sets should be valid for them. Let's formulate and prove analogical theorems for α-weak operations.

Theorems of idempotency.

Theorem 1. Operation of α-weak association is idempotent, that is

$$\tilde{A} \overset{\alpha}{\cup} \tilde{A} \overset{\alpha}{=} \tilde{A}.$$

Proof. Let's consider the fuzzy set $\tilde{C} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cup} \tilde{A}$ in X. According to the definition of the operation of α-weak association for an arbitrary element $x \in X$, we can write $\mu_{\tilde{A}}(x) \geq \alpha \vee \mu_{\tilde{A}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha$. Since the logical operation is idempotent, that is $\vee$, then for an arbitrary element $x \in X$, it will be fair $\mu_{\tilde{C}}(x) \geq \alpha \vee \mu_{\tilde{A}}(x) \geq \alpha$, what had to be proved.

It follows from the theorem 1, that the operation of weak association of fuzzy sets is also idempotent, that is

$$\tilde{A} \overset{0}{\cup} \tilde{A} \overset{0}{=} \tilde{A}.$$

By means of analogical considerations we can prove that the operations of α-weak and weak crossing are idempotent as well, that is

$$\tilde{A} \overset{\alpha}{\cap} \tilde{A} \overset{\alpha}{=} \tilde{A}.$$

$$\tilde{A} \overset{0}{\cap} \tilde{A} \overset{0}{=} \tilde{A}.$$

Theorems of distributivenes.

Theorem 2. Operations of α-weak crossing of fuzzy sets is distributive, that is

$$\tilde{A} \overset{\alpha}{\cap} \left(\tilde{B} \overset{\alpha}{\cup} \tilde{C}\right) \overset{\alpha}{=} \left(\tilde{A} \overset{\alpha}{\cap} \tilde{B}\right) \overset{\alpha}{\cup} \left(\tilde{A} \overset{\alpha}{\cap} \tilde{C}\right).$$

Proof. Let's consider $\tilde{C1} \overset{\alpha}{=} \tilde{B} \overset{\alpha}{\cup} \tilde{C}$, $\tilde{D1} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cap} \tilde{C1}$, $\tilde{C2} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cap} \tilde{B}$, $\tilde{C3} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cap} \tilde{C}$, $\tilde{D2} \overset{\alpha}{=} \tilde{C2} \overset{\alpha}{\cup} \tilde{C3}$. According to the definitions of the α-weak association and crossing operations for an arbitrary element $x \in X$ we can write

$$\mu_{\tilde{C1}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{B}}(x) \geq \alpha \vee \mu_{\tilde{C}}(x) \geq \alpha, \tag{6}$$

$$\mu_{\tilde{D}1}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_{\tilde{C}1}(x) \geq \alpha, \tag{7}$$

$$\mu_{\tilde{C}2}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_{\tilde{B}}(x) \geq \alpha, \tag{8}$$

$$\mu_{\tilde{C}3}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_{\tilde{C}}(x) \geq \alpha, \tag{9}$$

$$\mu_{\tilde{D}2}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{C}2}(x) \geq \alpha \vee \mu_{\tilde{C}3}(x) \geq \alpha. \tag{10}$$

Having done the substitution of the equivalent expressions for the logical variables $\mu_{\tilde{C}1}(x) \geq \alpha$, $\mu_{\tilde{C}2}(x) \geq \alpha$ and $\mu_{\tilde{C}3}(x) \geq \alpha$ from logical equations (6, 8, 9) into logical equations (7, 10) we obtain

$$\mu_{\tilde{D}1}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \wedge \left(\mu_{\tilde{B}}(x) \geq \alpha \vee \mu_{\tilde{C}}(x) \geq \alpha\right),$$

$$\mu_{\tilde{D}2}(x) \geq \alpha \Leftrightarrow \left(\mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_{\tilde{B}}(x) \geq \alpha\right) \vee \left(\mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_{\tilde{C}}(x) \geq \alpha\right).$$

Since logical operation $\wedge$ is distributive, that for an arbitrary element $x \in X$ we can claim, that $\mu_{\tilde{D}1}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{D}2}(x) \geq \alpha$, what had to be proved.

The operation of weak crossing of fuzzy sets is also distributive, that is

$$\tilde{A} \overset{0}{\cap} \left(\tilde{B} \overset{0}{\cup} \tilde{C}\right) \overset{0}{=} \left(\tilde{A} \overset{0}{\cap} \tilde{B}\right) \overset{0}{\cup} \left(\tilde{A} \overset{0}{\cap} \tilde{C}\right).$$

By means of analogical considerations we can prove that operations of α-weak and weak association are also distributive, that is

$$\tilde{A} \overset{\alpha}{\cup} \left(\tilde{B} \overset{\alpha}{\cap} \tilde{C}\right) \overset{\alpha}{=} \left(\tilde{A} \overset{\alpha}{\cup} \tilde{B}\right) \overset{\alpha}{\cap} \left(\tilde{A} \overset{\alpha}{\cup} \tilde{C}\right),$$

$$\tilde{A} \overset{0}{\cup} \left(\tilde{B} \overset{0}{\cap} \tilde{C}\right) \overset{0}{=} \left(\tilde{A} \overset{0}{\cup} \tilde{B}\right) \overset{0}{\cap} \left(\tilde{A} \overset{0}{\cup} \tilde{C}\right).$$

Theorems of involution.

Theorem 3: For any fuzzy set $\tilde{A}$ in X, the α-weak complement of its α-weak complement is α-weakly equal to the fuzzy set $\tilde{A}$, that is

$$\overset{\overset{\alpha}{\overline{\alpha}}}{\overline{\tilde{A}}} \overset{\alpha}{=} \tilde{A}.$$

Proof. Let's consider fuzzy sets $\tilde{B} \overset{\alpha}{=} \overset{\alpha}{\overline{\tilde{A}}}$ and $\tilde{C} \overset{\alpha}{=} \overset{\alpha}{\overline{\tilde{B}}}$ in X. According to the definition of α-weak complement, for the arbitrary element $x \in X$ we can write $\mu_{\tilde{B}}(x) < \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha$ and $\mu_{\tilde{C}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{B}}(x) < \alpha$. So, for an arbitrary element $x \in X$ the equivalency $\mu_{\tilde{C}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha$ will be fair, what had to be proved.

It follows from Theorem 3, that for any of fuzzy sets $\tilde{A}$ in X, the weak complement of its weak complement is weakly equal to the fuzzy set $\tilde{A}$, that is

$$\overset{\overset{0}{\overline{0}}}{\overline{\tilde{A}}} \overset{0}{=} \tilde{A}.$$

Theorems de Morgan.

Theorem 4. α-weak complement of the α-weak association of the fuzzy sets $\tilde{A}$ and $\tilde{B}$ in X are α-weakly equals to α-weak crossing of α-weak complement of these fuzzy sets, that is

$$\overset{\overset{\alpha}{\overline{\alpha}}}{\overline{\left(\tilde{A} \cup \tilde{B}\right)}} \overset{\alpha}{=} \overset{\alpha}{\overline{\tilde{A}}} \overset{\alpha}{\cap} \overset{\alpha}{\overline{\tilde{B}}}.$$

Proof. Let's consider fuzzy sets $\tilde{C}1 \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cup} \tilde{B}$, and $\tilde{C}2 \overset{\alpha}{=} \overset{\alpha}{\overline{\tilde{A}}} \overset{\alpha}{\cap} \overset{\alpha}{\overline{\tilde{B}}}$ and $\tilde{C}3 \overset{\alpha}{=} \overset{\alpha}{\overline{\tilde{C}1}}$ in X. According to the definitions of the corresponding operations, for the arbitrary element $x \in X$ we can write

$$\mu_{\tilde{C}1}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \vee \mu_{\tilde{B}}(x) \geq \alpha, \tag{11}$$

$$\mu_{\tilde{C}2}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) < \alpha \wedge \mu_{\tilde{B}}(x) < \alpha, \tag{12}$$

$$\mu_{\tilde{C}3}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{C}1}(x) < \alpha. \tag{13}$$

Taking into consideration that $\mu_{\tilde{C}1}(x) \geq \alpha \Leftrightarrow \neg\mu_{\tilde{C}1}(x) \geq \alpha$, let's do the substitution of the equivalent expression for the logical variable $\mu_{\tilde{C}1}(x) \geq \alpha$ from logical equation (11) into logical equation (13), and as a result we'll obtain

$$\mu_{\tilde{C}3}(x) \geq \alpha \Leftrightarrow \neg\left(\mu_{\tilde{A}}(x) \geq \alpha \vee \mu_{\tilde{B}}(x) \geq \alpha\right). \tag{14}$$

Since $\mu_{\tilde{A}}(x) < \alpha \Leftrightarrow \neg\left(\mu_{\tilde{A}}(x) \geq \alpha\right)$ and $\mu_{\tilde{B}}(x) < \alpha \Leftrightarrow \neg\left(\mu_{\tilde{B}}(x) \geq \alpha\right)$, the expression (12) we can write as

$$\mu_{\tilde{C}2}(x) \geq \alpha \Leftrightarrow \neg\left(\mu_{\tilde{A}}(x) \geq \alpha\right) \wedge \neg\left(\mu_{\tilde{B}}(x) \geq \alpha\right). \tag{15}$$

As it follows from the similar logical de Morgan's law

$$\neg\left(\mu_{\tilde{A}}(x) \geq \alpha \vee \mu_{\tilde{B}}(x) \geq \alpha\right) \Leftrightarrow \neg\left(\mu_{\tilde{A}}(x) \geq \alpha\right) \wedge \neg\left(\mu_{\tilde{B}}(x) \geq \alpha\right),$$

and the expressions (14) and (15) we can write $\mu_{\tilde{C}3}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{C}2}(x) \geq \alpha$, what had to be proved.

It follows from the theorem 4 that the weak complement of the weak association of fuzzy sets $\tilde{A}$ and $\tilde{B}$ in X weakly equals to the weak crossing of the weak complement of these fuzzy sets, that is

$$\left(\tilde{A} \overset{\frac{0}{0}}{\cup} \tilde{B}\right) \overset{0}{=} \overset{0}{\overline{\tilde{A}}} \overset{0}{\cap} \overset{0}{\overline{\tilde{B}}}.$$

By means of similar considerations we can prove the fairness of the second de Morgan' theorem for α-weak and weak operations, namely

$$\left(\tilde{A} \overset{\frac{\alpha}{\alpha}}{\cap} \tilde{B}\right) \overset{\alpha}{=} \overset{\alpha}{\overline{\tilde{A}}} \overset{\alpha}{\cup} \overset{\alpha}{\overline{\tilde{B}}},$$

$$\left(\tilde{A} \overset{\frac{0}{0}}{\cap} \tilde{B}\right) \overset{0}{=} \overset{0}{\overline{\tilde{A}}} \overset{0}{\cup} \overset{0}{\overline{\tilde{B}}}.$$

Besides above mentioned theorems, in classical theory of sets there are also theorems characterizing the operations between fuzzy sets and universum or empty set. Let's check the reality of the similar theorem for α-weak operations' class.

Theorem 5. α-weak association of the fuzzy set $\tilde{A}$ in X and the empty set $\oslash$ α-weakly equals to the fuzzy set $\tilde{A}$ in X, that is

$$\tilde{A} \overset{\alpha}{\cup} \oslash \overset{\alpha}{=} \tilde{A}.$$

Proof. Let's consider fuzzy set $\tilde{B} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cup} \oslash$ in X. According to the definition of α-weak association operation, for the arbitrary element $x \in X$ we can write $\mu_{\tilde{B}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \vee \mu_{\oslash}(x) \geq \alpha$. Since the definition of an empty set $\oslash$ $\mu_{\oslash}(x) = 0$, then $\mu_{\tilde{A}}(x) \geq \alpha \vee \mu_{\oslash}(x) = 0 \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha$. So, $\mu_{\tilde{B}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{B}}(x) \geq \alpha$, what had to be proved.

Similarly, the weak association of fuzzy set $\tilde{A}$ in X and the empty set $\oslash$ are weakly equals to the fuzzy set $\tilde{A}$ in X, that is

$$\tilde{A} \overset{0}{\cup} \oslash \overset{0}{=} \tilde{A}.$$

Theorem 6. α-weak crossing of the fuzzy set $\tilde{A}$ in X and the empty set $\oslash$ is α-weakly equal to the empty set $\oslash$, that is

$$\tilde{A} \overset{\alpha}{\cap} \oslash \overset{\alpha}{=} \oslash.$$

Proof. Let's consider the fuzzy set $\tilde{B} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cap} \oslash$ in X. According to the definition of the α-weak crossing operation, for the arbitrary element $x \in X$ we can write $\mu_{\tilde{B}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_{\oslash}(x) \geq \alpha$. As to the definition of the empty $\oslash$, that $\mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_{\oslash}(x) = 0 \Leftrightarrow \mu_{\oslash}(x) = 0$. So, $\mu_{\tilde{B}}(x) \geq \alpha \Leftrightarrow \mu_{\oslash}(x) = 0$, what had to be proved.

Similarly, a weak crossing of the fuzzy set $\tilde{A}$ in X and the empty set $\oslash$ weakly equals the empty set $\oslash$, that is

$$\tilde{A} \overset{0}{\cap} \oslash \overset{0}{=} \oslash.$$

Theorem 7. α-weak association of the fuzzy set $\tilde{A}$ in X with the universum X α-weakly equals to the universum X, that is

$$\tilde{A} \overset{\alpha}{\cup} X \overset{\alpha}{=} X.$$

Proof. Let's consider the fuzzy set $\tilde{B} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cup} X$ in X. Acccording to the definition of the α-weak association operation, for an arbitrary element $x \in X$ we can write $\mu_{\tilde{B}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \vee \mu_X(x) = 1$. As to the definition of the universum for all of the $x \in X$ $\mu_X(x) = 1$, that $\mu_{\tilde{A}}(x) \geq \alpha \vee \mu_X(x) = 1 \Leftrightarrow \mu_X(x) = 1$. So, $\mu_{\tilde{B}}(x) \geq \alpha \Leftrightarrow \mu_X(x) = 1$, what had to be proved.

Similarly, weak association of the fuzzy set $\tilde{A}$ in X with the universum X weakly equal to the universum X, that is

$$\tilde{A} \overset{0}{\cup} X \overset{0}{=} X.$$

Theorem 8. α-weak crossing of the fuzzy set $\tilde{A}$ in X with the universum X α-weakly equals the fuzzy set $\tilde{A}$ in X, that is

$$\tilde{A} \overset{\alpha}{\cap} X \overset{\alpha}{=} \tilde{A}.$$

Proof. Let's consider the fuzzy set $\tilde{B} \overset{\alpha}{=} \tilde{A} \overset{\alpha}{\cap} X$ is X. According to the definition of the α-weak crossing operation for an arbitrary element $x \in X$, we can write $\mu_{\tilde{B}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_X(x) \geq \alpha$. As to the definition of universum, for all $x \in X$ $\mu_X = 1$, that $\mu_{\tilde{A}}(x) \geq \alpha \wedge \mu_X(x) = 1 \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha$. So, $\mu_{\tilde{B}}(x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x) \geq \alpha$, what had to be proved. Similarly, the weak crossing of the fuzzy set $\tilde{A}$ in X with universum X weakly equals the fuzzy set $\tilde{A}$ in X, that is

$$\tilde{A} \overset{0}{\cap} X \overset{0}{=} \tilde{A}.$$

Let's consider the theorems characterizing α-weak operations between fuzzy sets and their α-weak complement. There are theorems for the Cantor sets

$$A \cup \bar{A} = X,$$

$$A \cap \bar{A} = \oslash.$$

In the traditional theory of fuzzy sets similar theorems are absent.

As for weak operations between fuzzy sets, the following theorem exists.

Theorem 9. Weak crossing of the fuzzy set $\tilde{A}$ in X with its weak complement $\overset{0}{\bar{\tilde{A}}}$ in X weakly equals the empty set $\oslash$, that is

$$\tilde{A} \overset{0}{\cap} \overset{0}{\bar{\tilde{A}}} \overset{0}{=} \oslash.$$

Proof. Let's consider fuzzy sets $\tilde{B} \overset{0}{=} \overset{0}{\bar{\tilde{A}}}$ and $\tilde{C} \overset{0}{=} \tilde{A} \overset{0}{\cap} \tilde{B}$ in X. According to the definition of α-weak crossing operation, for the arbitrary element $x \in X$ we can write

$$\mu_{\tilde{B}}(x) > 0 \Leftrightarrow \mu_{\tilde{A}}(x) = 0, \tag{16}$$

$$\mu_{\tilde{C}}(x) > 0 \Leftrightarrow \mu_{\tilde{A}}(x) > 0 \wedge \mu_{\tilde{B}}(x) > 0. \tag{17}$$

Having done the substitution of the equivalent expression for a logical variable $\mu_{\tilde{B}}(x) > 0$ from the logical equation (16) into the logical equation (17) we get $\mu_{\tilde{C}}(x) > 0 \Leftrightarrow \mu_{\tilde{A}}(x) > 0 \wedge \mu_{\tilde{A}}(x) = 0$.

Since $\mu_{\tilde{A}}(x) > 0 \wedge \mu_{\tilde{A}}(x) = 0 \Leftrightarrow False$, then $\mu_{\tilde{C}}(x) = 0$, what had to be proved.

Let's consider α-weak operations on binary fuzzy relations (BFR). Binary fuzzy relation $\left(\tilde{A}, X\right)$ – is a fuzzy set defined on the Descartes square $X \times X$ and for which the following is true:

$$\forall x, y \in X \left(\mu_{\tilde{A}}(x, y) \in [0, 1]\right).$$

Since BFR is a common fuzzy set and the only difference is that its elements are the ordered pairs of the Descartes square of the universum X, then for BFR all introduced beforehand α-weak operations occur (association, crossing, complement, difference etc). At the same time, for BFR one can introduce additionally operations which are absent for ordinary fuzzy sets. Therefore there is an inverted relation, its definition is in the traditional theory is written as:

$$\left(\tilde{A}^{-1}, X\right) \textit{ is the inverted relation to } \left(\tilde{A}, X\right) \textit{ if and only if, when}$$

$$\forall x, y \in X \left(\mu_{\tilde{A}^{-1}}(y, x) = \mu_{\tilde{A}}(x, y)\right).$$

Following the principles of building the class of weak operations, for the α-weak inverted relation we can write:

$$\left(\overset{\alpha}{\tilde{A}^{-1}}, X\right) \textit{is } \alpha - \textit{weak inverted relation to } \left(\tilde{A}, X\right) \textit{ if and only if, when}$$

$$\forall x, y \in X \left(\mu_{\tilde{A}^{-1}}(y, x) \geq \alpha \Leftrightarrow \mu_{\tilde{A}}(x, y) > 0\right).$$

Accordingly,

$$\left(\overset{0}{\tilde{A}^{-1}}, X\right) \textit{is } \alpha - \textit{weak inverted relation to } \left(\tilde{A}, X\right) \textit{ if and only if, when}$$

$$\forall x, y \in X \left(\mu_{\tilde{A}^{-1}}(y, x) > 0 \Leftrightarrow \mu_{\tilde{A}}(x, y) > 0\right).$$

Let's formulate the definition for a weak composition of fuzzy relations. Traditional maximin composition of fuzzy relations is formulated as: fuzzy relation $\left(\tilde{A}_1 \circ \tilde{A}_2, X\right)$ is a maximin composition of fuzzy relations $\left(\tilde{A}_1, X\right)$ and $\left(\tilde{A}_2, X\right)$ as to the definition if and only if the, when

$$\forall x, y \in X \left(\mu_{\tilde{A}_1 \circ \tilde{A}_2}(x, y) = \underset{z \in X}{MaxMin}(\mu_{\tilde{A}_1}(x, z), \mu_{\tilde{A}_2}(z, y))\right).$$

The definition for the α-weak composition can be written as: fuzzy relation $\left(\tilde{A}_1 \overset{\alpha}{\circ} \tilde{A}_2, X\right)$ is the α-weak composition of fuzzy relations $\left(\tilde{A}_1, X\right)$ and $\left(\tilde{A}_2, X\right)$ according to its definition if and only if, when

$$\forall x, y \in X \left(\mu_{\tilde{A}_1 \circ \tilde{A}_2}(x, y) \geq \alpha \Leftrightarrow \exists z \in X(\mu_{\tilde{A}_1}(x, z) \geq \alpha \wedge \mu_{\tilde{A}_2}(z, y)) \geq \alpha\right), \alpha \in (0, 1]. \tag{18}$$

It follows from (18) that fuzzy relation $\left(\tilde{A}_1 \overset{0}{\circ} \tilde{A}_2, X\right)$ is the α-weak composition of fuzzy relations $\left(\tilde{A}_1, X\right)$ and $\left(\tilde{A}_2, X\right)$ if and only if, when

$$\forall x, y \in X \left(\mu_{\tilde{A}_1 \circ \tilde{A}_2}(x, y) > 0 \Leftrightarrow \exists z \in X(\mu_{\tilde{A}_1}(x, z) > 0 \wedge \mu_{\tilde{A}_2}(z, y)) > 0\right), \alpha \in (0, 1].$$

Let's proceed to the fuzzy sets reflections and the extension principles. As it is known the extension principles is the way of defining the image of fuzzy set under crisp or fuzzy reflection. There can be many such methods, but all of them must satisfy two conditions:

1. The image of any fuzzy set, regardless of the nature of the reflection, is also a fuzzy set.

2. Any extension principle should not contradict the definition of a clear representation of classical Cantor sets.

The definition of the maximin of extension principle, the most widespread in the traditional theory of fuzzy sets, for the crisp reflection of fuzzy sets can be formulated as follows: fuzzy set $f(\tilde{A})$ in Y is the image of the fuzzy set $\tilde{A}$ in X under crisp reflection $f : X \to Y$ according to the definition if and only if, when

$$\forall y \in Y \left(\mu_{f(\tilde{A})}(y) = \underset{x \in f^{-1}(y)}{Max} \mu_{\tilde{A}}(x)\right), \tag{19}$$

where $f^{-1}(y)$ is the proimage of the element $y \in Y$ under crisp reflection $f : X \to Y$.

Maximin of extension principle for fuzzy reflection of the fuzzy sets one can be written as: fuzzy set $\tilde{f}(\tilde{A})$ in Y is the image of the fuzzy set $\tilde{A}$ in X under fuzzy reflection $f : X \to Y$ according to the definition if and only if, when

$$\forall y \in Y \left(\mu_{\tilde{f}(\tilde{A})}(y) = \underset{x \in X}{MaxMin}(\mu_{\tilde{A}}(x), \mu_{\tilde{f}}(x, y))\right), \tag{20}$$

where $\mu_{\tilde{f}} : X \times Y \to (0, 1]$ - membership function of fuzzy reflection $f : X \to Y$.

Let's formulate the extension principles for crisp and fuzzy reflections of fuzzy sets that are more general than (19, 20) and less demanding on the completeness of data on membership functions.

The definition of α-weak extension principle for crisp reflections of fuzzy sets is formulated as: fuzzy set $f(\tilde{A})$ in Y is the α-weak image of fuzzy set $\tilde{A}$ in X under crisp reflection $f : X \to Y$ according to the definition if and only if, when

$$\forall y \in Y \left(\mu_{f(\tilde{A})}(y) \geq \alpha \Leftrightarrow \exists x \in f^{-1}(y)(\mu_{\tilde{A}}(x) \geq \alpha)\right),$$

where $f^{-1}(y)$ is the proimage of the element $y \in Y$ under crisp reflection $f : X \to Y$.

Accordingly, for the principle of weak extension for crisp reflections of fuzzy sets we can write: fuzzy set $f(\tilde{A})$ in Y is a weak image of fuzzy set $\tilde{A}$ in X under crisp reflection $f : X \to Y$ according to the definition if and only if, when

$$\forall y \in Y \left(\mu_{f(\tilde{A})}(y) > 0 \Leftrightarrow \exists x \in f^{-1}(y)(\mu_{\tilde{A}}(x) > 0)\right).$$

The definition of α-weak extension principle for fuzzy reflections of fuzzy sets can be written as: fuzzy set $\tilde{f}(\tilde{A})$ in Y is the α-weak image of fuzzy set $\tilde{A}$ in X under fuzzy reflection $\tilde{f} : X \to Y$ according to the definition if and only if, when

$$\forall y \in Y \left(\mu_{\tilde{f}(\tilde{A})}(y) \geq \alpha \Leftrightarrow \exists x \in X(\mu_{\tilde{A}}(x) \geq \alpha) \wedge \mu_{\tilde{f}}(x, y) \geq \alpha\right),$$

where $\mu_{\tilde{f}} : X \times Y \to (0, 1]$ - membership function of fuzzy reflection $\tilde{f} : X \to Y$.

Accordingly, for the principle of weak extension for the fuzzy reflections of fuzzy sets we can write: fuzzy set $\tilde{f}(\tilde{A})$ in Y is a weak image of fuzzy set $\tilde{A}$ in X under fuzzy reflection $\tilde{f} : X \to Y$ according to the definition if and only if, when

$$\forall y \in Y \left(\mu_{\tilde{f}(\tilde{A})}(y) > 0 \Leftrightarrow \exists x \in X(\mu_{\tilde{A}}(x) > 0) \wedge \mu_{\tilde{f}}(x, y) > 0 \right).$$

Conclusions

1. There is a large number of applied problems for which the use of the maximin extension principle hinders their solution, since its application requires complete information about the membership functions of vaguely defined parameters of the problem, and this is often a practically impossible procedure. In these cases, even the highest-level expert can determine only cores or α-cuts for the unknown fuzzy parameters of the system. Building complete membership functions of unknown fuzzy parameters on this basis is risky and unreliable.

2. The axiomatics of the theory of fuzzy sets have been extended in order to introduce non-traditional (less demanding on the completeness of data on membership functions) extension principles and operations on fuzzy sets. The so-called α-weak operations on fuzzy sets are proposed, which are based on the use of α-cuts.

3. The axiomatics of weak operations is constructed so that each of these operations reduces to the corresponding classical operation in the case of degeneracy of fuzzy sets to classical Cantor sets.

4. For weak operations on fuzzy sets, the same theorems as for classical sets are valid, namely, theorems of idempotency, distributivity, involution, de Morgan and others.

5. Weak operations are introduced not only for fuzzy sets, but also for binary fuzzy relations, which made it possible to construct the principles of weak extension. All this makes it possible to use the mathematical apparatus of fuzzy sets to solve problems in conditions of significant uncertainty of input information.

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Анық емес жиындар теориясының дамуы: әлсіз операциялар және жалпылау принциптері

Жұмыста қолданбалы есептерді шешу үшін анық емес жиындар теориясын пайдалану кезінде туындайтын мәселелер қарастырылған. Статистикалық деректерге негізделген стохастикалық әдістерден айырмашылығы, статистикалық деректер болмаған кезде анық емес жиындар теориясы әдістерін қолданған жөн. Бұл жағдайларда алгоритмдер осы білім саласындағы мамандар болып табылатын сарапшылар жасаған тиістілік функциясына негізделуі керек. Ең дұрысы, тиістілік функциялары туралы толық ақпарат қажет, бірақ бұл практикалық процедура емес. Көбінесе, тіпті ең тәжірибелі маман тек олардың тасымалдаушыларын немесе белгісіз бұлыңғыр жүйе параметрлері үшін α-деңгейінің бөлек жиынтықтарын анықтай алады. Осы негізде белгісіз анық емес параметрлердің толық тиістілік функцияларын құру тәуекелді және сенімсіз. Сондықтан мақалада анық емес жиындар теориясының аксиоматикасын кеңейту ұсынылады (тиістілік функциялар туралы деректердің толықтығын талап етпейтін) жалпылаудың және анық емес жиындардағы операциялардың принциптерін енгізу. Бөлек α-деңгейлі жиындарды қолдануға негізделген анық емес жиындардағы α-әлсіз деп аталатын амалдар ұсынылған. Сондай-ақ, кеңейтілген аксиоматикалық теорияда Кантордың жиындар теориясының барлық классикалық теоремаларын қолдануға болатыны көрсетілген.

Ақпараттың маңызды белгісіздігі жағдайында мәселелерді шешуге мүмкіндік беретін жаңа жалпылау принциптері енгізілді.

Кілт сөздер: Кантор жиыны, анық емес жиын, тиістілік функция, α-деңгейлі жиын, анық емес жиынды қолдау, α-әлсіз операция.

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Развитие теории нечетких множеств: слабые операции и принципы обобщения

В работе рассмотрены проблемы, возникающие при использовании теории нечетких множеств для решения прикладных задач. В отличие от стохастических методов, основанных на статистических данных, методы теории нечетких множеств целесообразно применять, когда статистические данные недоступны. В этих случаях алгоритмы должны основываться на функциях принадлежности, формируемых экспертами, являющимися специалистами в данной области знаний. В идеале требуется полная информация о функциях принадлежности, но это непрактичная процедура. Чаще всего даже самый опытный специалист может определить только их носители или отдельные наборы α-уровня для неизвестных нечетких параметров системы. Построение на этой основе полных функций принадлежности неизвестных нечетких параметров рискованно и ненадежно. Поэтому в статье предложены расширение аксиоматики теории нечетких множеств с целью введения нетрадиционных (менее требовательных к полноте данных о функциях принадлежности) принципов обобщения и операций над нечеткими множествами, а также так называемые α-слабые операции над нечеткими множествами, основанные на использовании отдельных множеств α-уровня. Кроме того, показано, что все классические теоремы теории множеств Кантора применимы в расширенной аксиоматической теории. Введены новые принципы обобщения, позволяющие решать задачи в условиях значительной неопределенности информации.

Ключевые слова: множество Кантора, нечеткое множество, функция принадлежности, множество α-уровня, носитель нечеткого множества, α-слабая операция.