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ALMOST GEODESIC MAPPINGS OF TYPE π_1^* OF SPACES WITH AFFINE CONNECTION

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Summary. We consider almost geodesic mappings π_1^* of spaces with affine connections. This mappings are a special case of first type almost geodesic mappings. We have found the objects which are invariants of the mappings π_1^*. The fundamental equations of these mappings are in Cauchy form. We study π_1^* mappings of constant curvature spaces.

1 INTRODUCTION

In the theory of geodesic mappings and their generalizations many basic results were formulated as a system of differential equations in Cauchy form, see [1–14]. For almost geodesic mappings π_1 a similar result for special Ricci-Codazzi Riemannian spaces is formulated in Sinyukov monograph [1]. This result was generalized for Ricci-Codazzi spaces with affine connection and for Riemannian spaces in [15]. For π_1^* mappings of general symmetric spaces with affine connection the system of differential equations in the Cauchy form were found in works [16].

This paper is devoted to detailed study of π_1^* mappings which are characterized by the general equations in the Cauchy form. This result is significant because the equations in this form have established methods of solution.

The concept of almost geodesic mappings of type π_1^* of spaces with affine torsion-free connections was first introduced in [16]. These mappings are a special case of type π_1 almost geodesic mappings which were introduced by N. S. Sinyukov in [1].

The paper is devoted to study the general properties of π_1^* mappings. In particular, we have obtained the objects which are invariant under the mappings. Also π_1^* mappings of spaces of constant curvature and affine spaces were studied.

Let us recall the basic conceptions of the almost geodesic mappings theory presented in [1].

A curve defined in a space with an affine connection A_n is called *almost geodesic* if there exists a two-dimensional plane element parallel along the curve (relative to the affine connection) such that for any tangent vector of the curve its parallel translation along the curve belongs to the plane element.

A diffeomorphism f between spaces with affine connection A and $\overline{A}_n$ is called *almost geodesic mapping* if any geodesic curve of A is mapped under f onto an almost geodesic curve in $\overline{A}$.

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In order that a mapping of a space A_n onto a space $\overline{A}_n$ be almost geodesic it is necessary and sufficient that in a common coordinate system $x \equiv (x^1, x^2, \dots, x^n)$ which both spaces are referred to, the deformation tensor of the mapping $P_{ij}^h(x) \equiv \overline{\Gamma}_{ij}^h(x) - \Gamma_{ij}^h(x)$ must satisfy the conditions

$$A_{\alpha\beta\gamma}^h \lambda^\alpha \lambda^\beta \lambda^\gamma = a \cdot P_{\alpha\beta}^h \lambda^\alpha \lambda^\beta + b \cdot \lambda^h,$$

where $A_{ijk}^h \equiv P_{ij,k}^h + P_{ij}^\alpha P_{\alpha k}^h$, $\Gamma_{ij}^h(x)$ $\left(\overline{\Gamma}_{ij}^h(x)\right)$ are the components of the affine connection of the space A_n $\left(\overline{A}_n\right)$, $'',''$ denotes covariant derivative with respect to the connection of the space A_n, λ^h is an arbitrary vector, a and b are certain functions of variables x^h and λ^h.

Three types of almost geodesic mappings were specified, namely π_1, π_2, π_3. We have proved that for $n > 5$ other types of almost geodesic mappings except π_1, π_2, and π_3 do not exist [17].

Almost geodesic mappings of π_1 type are characterized by the following conditions for the deformation tensor:

$$A_{(ijk)}^h = \delta_{(i}^h a_{jk)} + b_{(i} P_{jk)}^h,$$

where a_{ij} is a certain symmetric tensor, b_i a certain covector, δ_i^h are the Kronecker delta, (ijk) denotes an operation called symmetrization without division with respect to the indices i, j and k.

Unlike mappings of the type π_1, the study of mappings π_2 and π_3 are devoted by a lot of papers (See e.g. [1,2]). It stems from the fact that the main equations of these mappings are much more sophisticated than equations of other ones. Hence the paper is devoted to a special case of mappings π_1, which does not degenerate into π_2, π_3 or geodesic mappings.

2 ALMOST GEODESIC MAPPINGS OF THE π_1^* TYPE

Let a mapping of A_n onto $\overline{A}_n$ satisfy the conditions [16]:

$$P_{ij,k}^h + P_{ij}^\alpha P_{\alpha k}^h = a_{ij} \delta_k^h, \tag{1}$$

where a_{ij} is a certain symmetric tensor.

These mappings are a special case of almost geodesic mappings of the π_1 type. From now on, that mappings will be denoted by π_1^*. Let us consider (1) as a system of differential equations of Cauchy type with respect to the deformation tensor P_{ij}^h and find their integrability conditions. To this end, differentiate covariantly (1) with respect to x^m in A_n, then alternate it in k and m.

Contracting the integrability conditions of the equations (1) for h and m, we get

$$(n-1) a_{ij,k} = P_{ij}^\alpha R_{\alpha k} - P_{\alpha(i}^\beta R_{j)\beta k}^\alpha - (n-1) P_{ij}^\alpha a_{\alpha k}, \tag{2}$$

where R_{ijk}^h is the Riemann tensor of the space A_n, $R_{ij} \equiv R_{ij\alpha}^\alpha$ is the Ricci tensor.

Obviously, in the space A_n the equations (1) and (2) form a closed system of PDEs of Cauchy type with respect to the functions $P_{ij}^h(x)$ and $a_{ij}(x)$. The functions must also satisfy the algebraic conditions

$$P_{ij}^h(x) = P_{ji}^h(x), \qquad a_{ij}(x) = a_{ji}(x). \tag{3}$$

Hence we have proved the theorem.

Theorem 1 *In order that a space A_n with an affine connection admits a canonical almost geodesic mapping of type π_1^* onto another space $\overline{A}_n$ with an affine connection, it is necessary and sufficient that the mixed system of differential equations of Cauchy type in covariant derivatives (1), (2), (3) has a solution with respect to the unknown functions $P_{ij}^h(x)$ and $a_{ij}(x)$.*

Let us note that Theorem 1 holds for $A_n \in C^1$ ($\Gamma_{ij}^h(x) \in C^1$), i.e. objects of affine connection Γ are differentiable. In this case, if $A_n \in C^r(r \geq 1)$ then $\overline{A}_n \in C^r$. It follows from the fact, that the solution $P_{ij}^h(x) \in C^r$ and $a_{ij}(x) \in C^{r-1}$.

The integrability conditions of the system are

$$-P_{ij}^\alpha R_{\alpha km}^h + P_{\alpha(i}^h R_{j)km}^\alpha = \frac{1}{n-1}[(P_{ij}^\alpha R_{\alpha m} - P_{\alpha(i}^\beta R_{j)m\beta}^\alpha)\delta_k^h - (P_{ij}^\alpha R_{\alpha k} - P_{\alpha(i}^\beta R_{j)k\beta}^\alpha)\delta_m^h,$$

where $[ij]$ denotes the alternation with respect to the mentioned indices.

3 INVARIANT OBJECTS UNDER π_1^* MAPPINGS

It is known [1], that if P_{ij}^h is a deformation tensor, then the Riemann tensors R_{ijk}^h and $\overline{R}_{ijk}^h$ of the spaces A_n and $\overline{A}_n$ are related to each other by the equations

$$\overline{R}_{ijk}^h = R_{ijk}^h + P_{i[k,j]}^h + P_{i[k}^\alpha P_{j]\alpha}^h. \tag{4}$$

Using the formulas (1) and (4), we get

$$*\overline{W}_{ijk}^h = *W_{ijk}^h, \tag{5}$$

where

$$*W_{ijk}^h \equiv R_{ijk}^h - \frac{1}{n-1}R_{i[j}\delta_{k]}^h, \quad *\overline{W}_{ijk}^h \equiv \overline{R}_{ijk}^h - \frac{1}{n-1}\overline{R}_{i[j}\delta_{k]}^h. \tag{6}$$

Obviously, $*W_{ijk}^h$ is a tensor of type (1,3) in the space A_n, and $*\overline{W}_{ijk}^h$ is a tensor of the same type in the space $\overline{A}_n$. From the relations (5) it follows that the tensor is invariant under almost geodesic mappings π_1^*.

Contracting (5) for h and i, it is easy to see that it holds

$$W_{ij} = \overline{W}_{ij}, \tag{7}$$

where

$$W_{ij} \equiv R_{[ij]}, \quad \overline{W}_{ij} \equiv \overline{R}_{[ij]}. \tag{8}$$

Taking account of (7), the formulas (5) are expressible in the form

$$\overline{W}_{ijk}^h = W_{ijk}^h, \tag{9}$$

where W_{ijk}^h and $\overline{W}_{ijk}^h$ are the Weyl tensors of projective curvature of the spaces A_n and $\overline{A}_n$ respectively.

Finally we obtained the theorem.

Theorem 2 *The Weyl tensor of projective curvature W^h_{ijk}, and also the tensors $*W^h_{ijk}$ and W_{ij} defined by the formulas (6) and (8) as geometric objects of spaces with affine connections are invariant under almost geodesic mappings of type π_1^*.*

4 MAPPINGS π_1^* OF EQUIAFFINE AND PROJECTIVE EUCLIDEAN SPACES

From the Theorem 2 we obtain the next one.

Theorem 3 *If a projective Euclidean space admits an almost geodesic mappings of type π_1^* onto $\overline{A}_n$, then $\overline{A}_n$ itself is also a projective Euclidean space.*

Theorem 4 *If an equiaffine space admits an almost geodesic mappings of type π_1^* onto $\overline{A}_n$, then $\overline{A}_n$ itself is also an equiaffine space.*

Proof. Obviously, the proof of Theorem 3 and 4 follows from the facts that the Weyl tensor of projective curvature vanishes in a projective Euclidean space, and for an equiaffine space the condition $W_{ij} = 0$ holds identically, respectively.

Hence because of Theorem 2 the above mentioned tensors vanish in the space $\overline{A}_n$. This means that $\overline{A}_n$ is a projective Euclidean and equiaffine space, respectively.

Thus from Theorem 3 and 4 projective Euclidean and equiaffine spaces form closed classes with respect to mappings of type π_1^*.

It is easy to see that the Riemann tensor is preserved under mappings π_1^* if and only if the tensor a_{ij} vanishes identically. In this case the main equations of the mappings become

$$P^h_{ij,k} = -P^\alpha_{ij} P^h_{\alpha k}. \tag{10}$$

In an affine space the equations (10) are completely integrable. Consequently, a solution of the equations is determined by arbitrary initial values of $P^h_{ij}(x_0)$. If the initial values satisfy the condition $P^h_{ij}(x_0) \not\equiv \delta^h_{(i}\psi_{j)}(x_0)$, then the constructed solution determines the mapping of an affine space A_n onto another affine space $\overline{A}_n$, and the mapping is different from a geodesic one.

Hence we obtain the theorem.

Theorem 5 *There is a mapping π_1^* of affine space onto itself such that all straight lines are mapped onto plain curves, and not all the curves are straight lines.*

Moreover, since in affine spaces the integrability conditions (2) of the equations (1) are satisfied identically, the equations (1) are completely integrable.

Let us prove the theorem.

Theorem 6 *Riemannian spaces V_n of non-zero constant curvature admit non-geodesic mappings π_1^*, which are also almost geodesic mappings of type π_3. The quadratic complex of geodesics is preserved under the mappings.*

Proof. Let V_n be a Riemannian spaces with non-zero constant curvature R which admits non-geodesic mappings π_1^*. The integrability conditions are expressible in the form

$$K\left(P^h_{k(i}g_{j)l} - P^h_{l(i}g_{j)k}\right) + \delta^h_l B_{ijk} - \delta^h_k B_{ijl} = 0, \tag{11}$$

where $B_{ijk} \equiv a_{ij,k} + P^\alpha_{ij}(a_{\alpha k} + K g_{\alpha k})$, g_{ij} is the metric tensor of the space V_n.

Let ϵ^h be a vector such that $\epsilon^\alpha \epsilon^\beta g_{\alpha\beta} = \pm 1$. Transvecting (11) with $\epsilon^j \epsilon^l$ and then symmetrizing it in i and k, we find

$$P^h_{ik} = \xi^h g_{ik} + \epsilon^h b_{ik} + \delta^h_{(i}\psi_{j)}, \tag{12}$$

where ξ^h, ψ_j are some vectors, b_{ik} is some symmetric tensor. From (12) it follows that the relation (11) becomes

$$\epsilon^h\left(b_{k(i}g_{j)l} - b_{l(i}g_{j)k}\right) + \delta^h_{[l}b_{k]ij} + \delta^h_{(i}g_{j)[l}\psi_{k]} = 0, \tag{13}$$

where b_{ijk} is some tensor.
Transvecting (13) with ϵ^l, we get

$$\delta^h_i\left(g_{j\alpha}\epsilon^\alpha\psi_k - g_{jk}\epsilon^\alpha\psi_\alpha\right) + \epsilon^h \underset{1}{b}_{ijk} + \delta^h_j \underset{2}{b}_{ik} + \delta^h_k \underset{3}{b}_{ij} = 0, \tag{14}$$

where $\underset{1}{b}_{ijk}$, $\underset{2}{b}_{ik}$, $\underset{3}{b}_{ij}$ are some tensors.

Suppose that $\psi_i \not\equiv 0$. Then $g_{j\alpha}\epsilon^\alpha\psi_k - g_{jk}\epsilon^\alpha\psi_\alpha \not\equiv 0$ and consequently there exist vectors a^j and b^k such that $a^j b^k\left(g_{j\alpha}\epsilon^\alpha\psi_k - g_{jk}\epsilon^\alpha\psi_\alpha\right) \neq 0$. Transvecting (13) with $a^j b^k$, we obtain a relation which is contrary to the assumption that $n > 3$. Hence $\psi_i = 0$. The formulas (14) can be simplified and we can show by a similar method that $b_{kij} = 0$. Then (13) becomes $b_{k(i}g_{j)l} - b_{l(i}g_{j)k} = 0$. Transvecting the latter with g^{jl}, we find that $b_{ki} = \frac{b_{\alpha\beta}g^{\alpha\beta}}{n} g_{ki}$. We have from (12) by direct calculation

$$P^h_{ij} = P^h g_{ij}, \tag{15}$$

where P^h is some vector. Hence the mapping is f-planar. Consequently, according to [1,2], such mapping is almost geodesic mapping of type π_3. And in [17] the authors have proved that the mappings $\pi_1 \cap \pi_3$ preserves the quadratic complex of geodesics [18].

Substituting (15) in (1), we have

$$P^h_{,k} + P^h P_k = \alpha \delta^h_k,$$

where α is some invariant, $P_k = P^i g_{ik}$.

Vector fields satisfying these conditions are referred to as *concircular vector fields*. One knows that concircular vector fields always exist in spaces of constant curvature.

5 EXAMPLES OF ALMOST GEODESIC MAPPINGS π_1^*

We shall give an example of an almost geodesic mapping π_1^* of a flat space A_n onto another flat space $\overline{A}_n$.

Let $x^1, x^2, \dots, x^n$ and $\overline{x}^1, \overline{x}^2, \dots, \overline{x}^n$ be affine coordinate systems in the spaces A_n and $\overline{A}_n$ respectively. A point mapping

$$\overline{x}^h = \tfrac{1}{2} C^h_\alpha (x^\alpha - C^\alpha)^2 + x^h_0, \tag{16}$$

where C_i^h, C^h, x_0^h are some constant, det $\| C_i^h \| \not\equiv 0$, defines the almost geodesic mapping π_1^* of the space A_n onto the space $\overline{A}_n$.

By direct calculation it is readily shown that the components of the deformation tensor P_{ij}^h in the coordinate system $x^1, x^2, \dots, x^n$ are given

$$P_{ii}^i = \frac{1}{x^i - C^i} \quad i = \overline{1, n},$$

all the other components being zero.

Obviously, the tensor satisfies the equation (10). Note that the mapping is different from mappings of types π_2 and π_3.

Straight lines which are defined in the space A_n by the equation $x^h = a^h + b^h t$ (t is a parameter along a line) are mapped into parabolas in the space $\overline{A}_n$. The parabolas are defined by the equations

$$\overline{x}^h = F^h + D^h t + E^h t^2,$$

where $F^h = \frac{1}{2} C_\alpha^h (a^\alpha - C^\alpha)^2$, $D^h = C_\alpha^h (a^\alpha - C^\alpha) b^\alpha$, $E^h = \frac{1}{2} C_\alpha^h (b^\alpha)^2$.

The exceptions are the straight lines through the point $M(C^1, C^2, \dots, C^n)$. By (16) the lines are mapped into straight lines too.

Finally we note that the formulas (16) generate a family of almost geodesic mappings π_1 of a flat space if the parameters C_i^h, C^h and x_0^h are understood as continuous values.

6 CONCLUSION

Out of the three types of almost geodesic mappings of spaces with affine connection, distinguished by N.S. Sinyukov, the least studied are almost geodesic mappings of the first type. The equations that characterize them are very complex. Therefore, the results obtained for mappings π_1^*, including for their particular cases, are very relevant and are of theoretical value from the geometrical point of view. At the same time, they can be used in the theory of relativity and theoretical mechanics.

Almost geodesic mappings are a natural generalization of geodesic mappings. The basic equations of geodesic mappings of spaces with affine connection cannot be reduced to closed systems of equations in covariant derivatives of Cauchy type, since the general solution depends on *n* arbitrary functions.

We have singled out a special case of almost geodesic mappings of the first type, denoted by π_1^*, the basic equations of which are reduced to a closed system of equations in covariant derivatives of the Cauchy type. This result is very important, since (since geodesic mappings are a special case of almost geodesic mappings) the basic equations of the first type of spaces with affine connection are not reducible to closed systems of equations in covariant derivatives of Cauchy type.

For the mappings π_1^* geometric objects of tensor nature are found that are invariant under such mappings. It turns out that the Weyl tensor is invariant not only with respect to geodesic mappings, but also with respect to more general mappings.

In the article it is proved that projective-Euclidean and equiaffine spaces form closed classes with respect to mappings π_1^*.

From geometrical point of view, an interesting is a special case of mappings π_1^*, which we have distinguished, where the Riemann tensor is invariant. In this case, the basic equations of such mappings in flat space are completely integrable. An example of mappings π_1^* of flat space onto flat space is given.

In wpresented paper, it is of interest to study the integrability conditions and their differential extensions of the obtained equations in covariant derivatives of the Cauchy type that characterize the mappings π_1^* of spaces with affine connection.

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