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ALMOST GEODESIC MAPPINGS OF THE SECOND TYPE OF SPACES WITH AFFINE CONNECTION ONTO TWO-SYMMETRIC SPACES

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Abstract. In the paper we consider canonical almost geodesic mappings of type $\pi_2(e)$ of spaces with affine connection onto two-symmetric spaces. The main equations for the mappings are obtained as a closed mixed system of PDEs of Cauchy type. We have found the maximum number of essential parameters which the solution of the system depends on.

Keywords: Almost geodesic mapping, spaces with affine connection, two-symmetric spaces

Mathematics subject classification: Primary 53B05; Secondary 53B99

1 Introduction

The paper is devoted to further study of the theory of almost geodesic mappings of affinely connected spaces. The theory goes back to the paper [16] of Levi-Civita, in which the problem on the search for Riemannian spaces with common geodesics was stated and solved in a special coordinate system. We note a remarkable fact that this problem is related to the study of equations of dynamics of mechanical systems.

The theory of geodesic mappings has been developed by T. Thomas, H. Weyl, P. A. Shirokov, A. S. Solodovnikov, N. S. Sinyukov, A. V. Aminova, J. Mikeš, and others, see [19, 23, 27].

Issues arisen by the exploration were studied by V. F. Kagan, G. Vrançeanu, Ya. L. Shapiro, D. V. Vedenyapin et al. The authors discover special classes of $(n-2)$-projective spaces.

In [24], Petrov introduced the notion of quasi-geodesic mappings. In particular, holomorphically projective mappings of Kählerian spaces are special quasi-geodesic mappings; they were examined by T. Otsuki and Y. Tashiro, M. Prvanović and others.

A natural generalization of these classes of mappings is the class of almost geodesic mappings introduced by Sinyukov (see [26, 27, 28]); he also specified three types of almost geodesic mappings π_1, π_2, π_3.

The types of almost geodesic mappings π_1, π_2, π_3 can intersect. The problem of completeness of classification had long remained unresolved. Berezovskii and Mikeš [4, 5] proved that for $n > 5$ other types of almost geodesic mappings except π_1, π_2, and π_3 do not exist.

The theory of almost geodesic mappings was developed by V. S. Sobchuk [30, 29], N. V. Yablonskaya [33], V. E. Berezovskii, J. Mikeš [1, 2, 3, 5, 6, 7, 8, 9, 10, 12, 11, 23, 21], Lj. S. Velimirović, N. Vesić, M. S. Stanković [32] et al. An interesting generalization is rotary mapping, see [22].

The paper is devoted to study canonical almost geodesic mappings of type $\pi_2(e)$ $(e = \pm 1)$ of spaces with affine connections onto two-symmetric spaces. The main equations for the mappings are obtained as a closed mixed system of PDEs of Cauchy type. We have found the maximum number of essential parameters which the general solution of the system depends on.

The investigations use local coordinates. We assume that all functions under consideration are sufficiently differentiable, n is the dimension of the studied spaces, and we suppose that $n > 2$.

2 Basic definitions of almost geodesic mappings of spaces with affine connections

Let us recall the basic definition, formulas and theorems of the theory presented in [9, 23, 21, 26, 27, 28].

Consider a space A_n with affine connection Γ^h_{ij} without torsion. The space is referred to coordinates $x^1, x^2, \dots, x^n$.

A curve $l\colon x^h = x^h(t)$ in the space A_n is a *geodesic* when its tangent vector $\lambda^h(t) = dx^h(t)/dt$ satisfies the equations

$$\lambda^h_1 = \rho(t) \cdot \lambda^h,$$

where

$$\lambda^h_1 \equiv \lambda^h_{,\alpha}\lambda^\alpha = \frac{d\lambda^h(t)}{dt} + \Gamma^h_{\alpha\beta}(x(t))\lambda^\alpha(t)\lambda^\beta(t),$$

and $\rho(t)$ is some function of t. We denote by comma " , " the covariant derivative with respect to the connection of the space A_n.

A curve in the space A_n is an *almost geodesic* when its tangent vector $\lambda^h(t) = dx^h(t)/dt$ satisfies the equations

$$\lambda^h_2 = a(t) \cdot \lambda^h + b(t) \cdot \lambda^h_1,$$

where $\lambda^h_2 \equiv \lambda^h_{1,\alpha}\lambda^\alpha$, and $a(t)$, $b(t)$ are some functions of the parameter t.

We say that a diffeomorphism $f\colon A_n \to \overline{A}_n$ is an *almost geodesic mapping* if any geodesic curve of A_n is mapped under f onto an almost geodesic curve in $\overline{A}_n$.

Suppose, that a space A_n with affine connection $\Gamma^h_{ij}(x)$ admits a mapping f onto space $\overline{A}_n$ with affine connection $\overline{\Gamma}^h_{ij}(x)$, and the spaces are referred to the common coordinate system $x^1, x^2, \dots, x^n$.

The tensor

$$P^h_{ij}(x) = \overline{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x), \tag{1}$$

is called a *deformation tensor* of the connections $\Gamma^h_{ij}(x)$ and $\overline{\Gamma}^h_{ij}(x)$ with respect to the mapping f. The symbols $\Gamma^h_{ij}(x)$ and $\overline{\Gamma}^h_{ij}(x)$ are components of affine connections of the spaces A_n and $\overline{A}_n$ respectively.

According to [27, 28], a necessary and sufficient condition for the mapping f of a space A_n onto a space $\overline{A}_n$ to be almost geodesic is that the deformation tensor $P^h_{ij}(x)$ of the mapping f in the common coordinate system $x^1, x^2, \dots, x^n$ has to satisfy the condition

$$A^h_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma = a \cdot P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + b \cdot \lambda^h,$$

where λ^h is an arbitrary vector, a and b are certain functions of variables $x^1, x^2, \dots, x^n$ and $\lambda^1, \lambda^2, \dots, \lambda^n$. The tensor A^h_{ijk} is defined as

$$A^h_{ijk} \overset{\text{def}}{=} P^h_{ij,k} + P^\alpha_{ij}P^h_{\alpha k}.$$

A mapping $f\colon A_n \to \overline{A}_n$ is called an *almost geodesic of type* π_2, if the conditions

$$P^h_{ij} = \delta^h_{(i}\psi_{j)} + F^h_{(i}\varphi_{j)}, \tag{2}$$

$$F^h_{(i,j)} + F^h_\alpha F^\alpha_{(i}\varphi_{j)} = \delta^h_{(i}\mu_{j)} + F^h_{(i}\rho_{j)}, \tag{3}$$

holds. Here ψ_i, φ_i, μ_i, ρ_i are some covectors, F^h_i is a tensor of type $(1,1)$, δ^h_i is the Kronecker delta and round bracket is denoted symmetrization.

We consider mappings $\pi_2\colon A_n \to \overline{A}_n$ characterized locally in a common coordinate system by the equations (2) and (3) as corresponding to $F^h_i(x)$.

A mapping π_2 satisfies a *mutuality condition* if the inverse mapping is also an almost geodesic of type π_2 and corresponding to the same affinor $F^h_i(x)$.

The mappings π_2 satisfying a *mutuality condition* will be denoted as $\pi_2(e)$, where $e = \pm 1, 0$, and for F the following identity holds:

$$F^2 = -eI. \tag{4}$$

As it was proved in [31], in the case when $e = \pm 1$ the basic equations of the mappings $\pi_2(e)$ can be written as the equations (2), (3) and (4):

$$F^h_{i,j} = F^h_{ij}, \qquad F^h_{ij,k} = \overset{6}{\Theta}{}^h_{ijk}, \qquad \mu_{i,j} = \mu_{ij}, \qquad \mu_{ij,k} = \overset{7}{\Theta}_{ijk}, \tag{5}$$

$$F^h_{(ij)} = F^h_{(i}\mu_{j)} - \delta^h_{(i}F^\alpha_{j)}\mu_\alpha, \qquad \mu_{(ij)} = \overset{5}{\Theta}_{ij}, \tag{6}$$

where $F^h_i(x)$, $F^h_{ij}(x)$, $\mu_i(x)$, $\mu_{ij}(x)$ are unknown functions. The above tensors are as follows:

$$\begin{aligned}
\overset{1}{\Theta}{}^h_{ijk} &\equiv \overset{2}{\Theta}{}^h_{ijk} + \overset{2}{\Theta}{}^h_{kji} - \overset{2}{\Theta}{}^h_{jki} + 2F^h_\alpha R^\alpha_{kji} - F^\alpha_i R^h_{\alpha jk} + F^\alpha_j R^h_{\alpha ik} + F^\alpha_k R^h_{\alpha ij},\\
\overset{2}{\Theta}{}^h_{ijk} &\equiv \mu_{(i}F^h_{j)k} - \delta^h_{(i}F^\alpha_{j)k}\mu_\alpha,\\
\overset{3}{\Theta}{}^h_{ijk} &\equiv \overset{2}{\Theta}{}^h_{ijk} - \overset{2}{\Theta}{}^h_{kji} + F^\alpha_j R^h_{\alpha ik} - F^h_\alpha R^\alpha_{jik},\\
\overset{4}{\Theta}_{jk} &\equiv F^\alpha_\beta \overset{1}{\Theta}{}^\beta_{\alpha jk} + 2F^\alpha_{\beta j}F^\beta_{\alpha k},\\
\overset{5}{\Theta}_{jk} &\equiv \frac{1}{(n-1-F^\alpha_\alpha)^2 - 1}\Big((n-1-F^\alpha_\alpha)\overset{4}{\Theta}_{ij} + \overset{4}{\Theta}_{\alpha\beta}F^\alpha_i F^\beta_j\Big),\\
\overset{6}{\Theta}{}^h_{ijk} &\equiv \frac{1}{2}\big(F^h_i\mu_{(jk)} + F^h_j\mu_{[ik]} + F^h_k\mu_{[ij]} - \delta^h_i m_{(jk)} - \delta^h_j m_{[ik]} - \delta^h_k m_{[ij]} + \overset{1}{\Theta}{}^h_{ikj}\big),\\
\overset{7}{\Theta}_{ijk} &\equiv \mu_\alpha R^\alpha_{kji} + \frac{1}{2}\big(\overset{5}{\Theta}_{ij,k} + \overset{5}{\Theta}_{ik,j} - \overset{5}{\Theta}_{jk,i}\big), \qquad m_{ij} \equiv F^\alpha_i\mu_{\alpha j},
\end{aligned}$$

where $R^h_{ijk}(x)$ is the Riemann tensor of the space A_n. We denote by the brackets $[i,k]$ an operation called antisymmetrization (or, alternation) without division with respect to the indices i and k.

Obviously, right hand sides of the equations (5) depend on unknown functions $F^h_i(x)$, $F^h_{ij}(x)$, $\mu_i(x)$, $\mu_{ij}(x)$, and on the components $\Gamma^h_{ij}(x)$ of the space A_n.

The equations (5) and (6) form a closed mixed system of PDE's of Cauchy type with respect to functions $F^h_i(x)$, $F^h_{ij}(x)$, $\mu_i(x)$, $\mu_{ij}(x)$. Also the mapping $\pi_2(e)$ depends on unknown functions $\psi_i(x)$, $\varphi_j(x)$ (see equations (2)).

An almost geodesic mapping π_2 for which $\psi_i \equiv 0$ is called *canonical*. It's known [28] that any almost geodesic mapping π_2 can be written as the composition of a canonical almost geodesic mapping of type π_2 and a geodesic mapping. The latter may be referred to as a trivial almost geodesic mapping.

Hence a canonical almost geodesic mapping $\pi_2(e)$ $(e = \pm 1)$, determined by the equations

$$P^h_{ij} = F^h_{(i}\varphi_{j)}, \tag{7}$$

and also by the equations (5) and (6).

3 Canonical almost geodesic mappings $\pi_2(\mathrm{e})$ $(\mathrm{e} = \pm 1)$ of spaces $\mathrm{A_n}$ with affine connection onto two-symmetric spaces

A space $\overline{A}_n$ with affine connection is called two-symmetric if its Riemann tensor satisfies the condition

$$\overline{R}^h_{ijk|ml} = 0,$$

where $\overline{R}^h_{ijk}$ is the Riemann tensor of the space $\overline{A}_n$. By the symbol "$|$" we denote a covariant derivative with respect to the connection of the space $\overline{A}_n$.

It is understood that in two-symmetric spaces $\overline{A}_n$ the conditions

$$\overline{R}^h_{ijk|m} \neq 0$$

are satisfied.

Then the two-symmetric spaces are other than symmetric spaces.

We recall that symmetric spaces $\overline{A}_n$ are characterized by

$$\overline{R}^h_{ijk|m} = 0.$$

Symmetric spaces were introduced by P.A. Shirokov [25] and É. Cartan [13], see also S. Helgason [14]. Geodesic mappings of this spaces were studied by N. S. Sinyukov [27], and I. Hinterleitner and J. Mikeš [15]. Geodesic mappings and holomorphically projective mappings of two-symmetric Riemannian spaces were stadied by J. Mikeš [18, 17], see [19, 20, 23, 27].

Let us consider the canonical almost geodesic mappings of type $\pi_2(e)$ $(e = \pm 1)$ of spaces A_n with affine connection onto two-symmetric spaces $\overline{A}_n$, which are determined by the equations (5), (6) and (7). Suppose, that the spaces are referred to the common coordinate system $x^1, x^2, \dots, x^n$.

Since

$$\overline{R}^h_{ijk|m} = \frac{\partial \overline{R}^h_{ijk}}{\partial x^m} + \overline{\Gamma}^h_{m\alpha}\overline{R}^\alpha_{ijk} - \overline{\Gamma}^\alpha_{mi}\overline{R}^h_{\alpha jk} - \overline{\Gamma}^\alpha_{mj}\overline{R}^h_{i\alpha k} - \overline{\Gamma}^\alpha_{mk}\overline{R}^h_{ij\alpha},$$

then taking account of (1) we can obtain

$$\overline{R}^h_{ijk|m} = \overline{R}^h_{ijk,m} + P^h_{m\alpha}\overline{R}^\alpha_{ijk} - P^\alpha_{mi}\overline{R}^h_{\alpha jk} - P^\alpha_{mj}\overline{R}^h_{i\alpha k} - P^\alpha_{mk}\overline{R}^h_{ij\alpha}. \tag{8}$$

Since according to the definition of covariant derivative

$$\begin{aligned}\left(\overline{R}^h_{ijk|m}\right)_{,l} = \frac{\partial \overline{R}^h_{ijk|m}}{\partial x^l} + \Gamma^h_{\alpha l}\overline{R}^\alpha_{ijk|m} - \Gamma^\alpha_{il}\overline{R}^h_{\alpha jk|m} - \Gamma^\alpha_{jl}\overline{R}^h_{i\alpha k|m} \\ -\Gamma^\alpha_{kl}\overline{R}^h_{ij\alpha|m} - \Gamma^\alpha_{ml}\overline{R}^h_{ijk|\alpha},\end{aligned}$$

then taking account of (1), we have

$$\begin{aligned}\left(\overline{R}^h_{ijk|m}\right)_{,l} = \overline{R}^h_{ijk|ml} - P^h_{\alpha l}\overline{R}^\alpha_{ijk|m} + P^\alpha_{il}\overline{R}^h_{\alpha jk|m} + P^\alpha_{jl}\overline{R}^h_{i\alpha k|m} \\ +P^\alpha_{kl}\overline{R}^h_{ij\alpha|m} + P^\alpha_{ml}\overline{R}^h_{ijk|\alpha}.\end{aligned} \tag{9}$$

Differentiating (8) with respect to x^l in the space A_n, we obtain

$$\begin{aligned}\left(\overline{R}^h_{ijk|m}\right)_{,l} = \overline{R}^h_{ijk,ml} + P^h_{m\alpha,l}\overline{R}^\alpha_{ijk} + P^h_{m\alpha}\overline{R}^\alpha_{ijk,l} - P^\alpha_{mi,l}\overline{R}^h_{\alpha jk} - P^\alpha_{mi}\overline{R}^h_{\alpha jk,l} \\ -P^\alpha_{mj,l}\overline{R}^h_{i\alpha k} - P^\alpha_{mj}\overline{R}^h_{i\alpha k,l} - P^\alpha_{mk,l}\overline{R}^h_{ij\alpha} - P^\alpha_{mk}\overline{R}^h_{ij\alpha,l}.\end{aligned} \tag{10}$$

Substituting in (9) from (10), we have

$$\begin{aligned}\overline{R}^h_{ijk,ml} = \overline{R}^h_{ijk|ml} - P^h_{\alpha l}\overline{R}^\alpha_{ijk|m} + P^\alpha_{il}\overline{R}^h_{\alpha jk|m} + P^\alpha_{jl}\overline{R}^h_{i\alpha k|m} + P^\alpha_{kl}\overline{R}^h_{ij\alpha|m} \\ +P^\alpha_{ml}\overline{R}^h_{ijk|\alpha} - P^h_{m\alpha,l}\overline{R}^\alpha_{ijk} - P^h_{m\alpha}\overline{R}^\alpha_{ijk,l} + P^\alpha_{mi,l}\overline{R}^h_{\alpha jk} + P^\alpha_{mi}\overline{R}^h_{\alpha jk,l} \\ +P^\alpha_{mj,l}\overline{R}^h_{i\alpha k} + P^\alpha_{mj}\overline{R}^h_{i\alpha k,l} + P^\alpha_{mk,l}\overline{R}^h_{ij\alpha} + P^\alpha_{mk}\overline{R}^h_{ij\alpha,l}.\end{aligned} \tag{11}$$

Suppose that the space $\overline{A}_n$ is a two-symmetric space. Then the formula (7) holds. Hence from (11) we obtain

$$\begin{aligned}\overline{R}^h_{ijk,ml} = -P^h_{\alpha l}\overline{R}^\alpha_{ijk|m} + P^\alpha_{il}\overline{R}^h_{\alpha jk|m} + P^\alpha_{jl}\overline{R}^h_{i\alpha k|m} + P^\alpha_{kl}\overline{R}^h_{ij\alpha|m} \\ +P^\alpha_{ml}\overline{R}^h_{ijk|\alpha} - P^h_{m\alpha,l}\overline{R}^\alpha_{ijk} - P^h_{m\alpha}\overline{R}^\alpha_{ijk,l} + P^\alpha_{mi,l}\overline{R}^h_{\alpha jk} + P^\alpha_{mi}\overline{R}^h_{\alpha jk,l} \\ +P^\alpha_{mj,l}\overline{R}^h_{i\alpha k} + P^\alpha_{mj}\overline{R}^h_{i\alpha k,l} + P^\alpha_{mk,l}\overline{R}^h_{ij\alpha} + P^\alpha_{mk}\overline{R}^h_{ij\alpha,l}.\end{aligned} \tag{12}$$

We introduce the tensor $\overline{R}^h_{ijkm}$ defined by

$$\overline{R}^h_{ijk,m} = \overline{R}^h_{ijkm}. \tag{13}$$

It's known [27, 28] that the Riemann tensors of the spaces A_n and $\overline{A}_n$ are related to each other by the equations

$$\overline{R}^h_{ijk} = R^h_{ijk} + P^h_{ik,j} - P^h_{ij,k} + P^\alpha_{ik}P^h_{\alpha j} - P^\alpha_{ij}P^h_{\alpha k}. \tag{14}$$

Since the deformation tensor of the mapping $P^h_{ij}(x)$ is represented by the equations (6), it follows from (14) that

$$\varphi_{i,j}F^h_k + \varphi_{k,j}F^h_i - \varphi_{i,k}F^h_j - \varphi_{j,k}F^h_i = C^h_{ijk}, \tag{15}$$

where

$$\begin{aligned} C^h_{ijk} = \overline{R}^h_{ijk} - R^h_{ijk} - \varphi_i\big(F^h_{kj} + \varphi_\alpha F^\alpha_k F^h_j + e\delta^h_k\varphi_j - F^h_{jk} - \varphi_\alpha F^\alpha_j F^h_k - e\delta^h_j\varphi_k\big) \\ +\varphi_k\big(F^h_{ij} + \varphi_\alpha F^\alpha_i F^h_j\big) - \varphi_j\big(F^h_{ik} + \varphi_\alpha F^\alpha_i F^h_k\big). \end{aligned}$$

Let us multiply (15) by F^m_l and contract for l and h. Hence we have

$$\delta^m_k\varphi_{i,j} + \delta^m_i\varphi_{k,j} - \delta^m_j\varphi_{i,k} - \delta^m_i\varphi_{j,k} = eC^\alpha_{ijk}F^m_\alpha. \tag{16}$$

Contracting the equations (16) for m and i we get

$$\varphi_{k,j} - \varphi_{j,k} = \frac{e}{n+1}C^\alpha_{\beta jk}F^\beta_\alpha. \tag{17}$$

Again, contracting the equations (16) for m and k we get

$$n\varphi_{i,j} - \varphi_{j,i} = eC^\alpha_{ij\beta}F^\beta_\alpha. \tag{18}$$

Taking account of (17) the equations (18) can be written as

$$\varphi_{i,j} = \frac{e}{n-1}\big(C^\alpha_{ij\beta} - \frac{1}{n+1}C^\alpha_{\beta ji}\big)F^\beta_\alpha. \tag{19}$$

And finally, taking account of (4), (6) and (13) the equations (12) can be written as

$$\begin{aligned} \overline{R}^h_{ijkm,l} = -F^h_{(\alpha}\varphi_{l)}\overline{R}^\alpha_{ijk|m} + F^\alpha_{(i}\varphi_{l)}\overline{R}^h_{\alpha jk|m} + F^\alpha_{(j}\varphi_{l)}\overline{R}^h_{i\alpha k|m} + F^\alpha_{(k}\varphi_{l)}\overline{R}^h_{ij\alpha|m} \\ +F^\alpha_{(m}\varphi_{l)}\overline{R}^h_{ijk|\alpha} - \big(F^h_{(m|l|}\varphi_{\alpha)} + F^h_{(m}\varphi_{\alpha),l}\big)\overline{R}^\alpha_{ijk} - F^h_{(m}\varphi_{\alpha)}\overline{R}^\alpha_{ijkl} \\ +\big(F^\alpha_{(m|l|}\varphi_{i)} + F^\alpha_{(m}\varphi_{i),l}\big)\overline{R}^h_{\alpha jk} + F^\alpha_{(m}\varphi_{i)}\overline{R}^h_{\alpha jkl} + \big(F^\alpha_{(m|l|}\varphi_{j)} + F^\alpha_{(m}\varphi_{j),l}\big)\overline{R}^h_{i\alpha k} \\ +F^\alpha_{(m}\varphi_{j)}\overline{R}^h_{i\alpha kl} + \big(F^\alpha_{(m|l|}\varphi_{k)} + F^\alpha_{(m}\varphi_{k),l}\big)\overline{R}^h_{ij\alpha} + F^\alpha_{(m}\varphi_{k)}\overline{R}^h_{ij\alpha l}. \end{aligned} \tag{20}$$

Suppose, that in the equations (20) the tensors $\overline{R}^h_{ijk|m}$ and $\varphi_{i,j}$ are expressed according to (8) and (19). Also we suppose that $\overline{R}^h_{ijk,m} = \overline{R}^h_{ijkm}$. Obviously, in the space A_n the equations (4), (13), (19) and (20) form a closed mixed system of PDE's of Cauchy type with respect to functions $F^h_i(x)$, $F^h_{ij}(x)$, $\mu_i(x)$, $\mu_{ij}(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijkm}(x)$, $\varphi_i(x)$, and the functions $F^h_i(x)$, $F^h_{ij}(x)$, $\mu_i(x)$, $\mu_{ij}(x)$ must satisfy the algebraic conditions (5). The algebraic conditions for the functions $\overline{R}^h_{ijk}(x)$ and $\overline{R}^h_{ijkm}(x)$ are

$$\begin{aligned} \overline{R}^h_{i(jk)} = 0, \quad \overline{R}^h_{(ijk)} = 0, \qquad \overline{R}^h_{i(jk)m} = 0, \quad \overline{R}^h_{(ijk)m} = 0, \\ \overline{R}^h_{i(jkm)} = -F^h_{(m}\varphi_{\alpha)}\overline{R}^\alpha_{ijk} - F^h_{(k}\varphi_{\alpha)}\overline{R}^\alpha_{imj} - F^h_{(j}\varphi_{\alpha)}\overline{R}^\alpha_{ikm} \\ + F^\alpha_{(m}\varphi_{i)}\overline{R}^h_{\alpha jk} + F^\alpha_{(k}\varphi_{i)}\overline{R}^h_{\alpha mj} + F^\alpha_{(j}\varphi_{i)}\overline{R}^h_{\alpha km}. \end{aligned} \tag{21}$$

Hense we have proved

Theorem. *In order that a space A_n with affine connection admits an almost geodesic mappings of type $\pi_2(e)$ ($e = \pm 1$) onto a two-symmetric space $\overline{A}_n$, it is necessary and sufficient that the* *mixed system of differential equations of Cauchy type in covariant derivatives* (5), (6), (13), (19), (20), (21) *has a solution with respect to functions* $F_i^h(x)$, $F_{ij}^h(x)$, $\mu_i(x)$, $\mu_{ij}(x)$, $\overline{R}_{ijk}^h(x)$, $\overline{R}_{ijkm}^h(x)$, $\varphi_i(x)$.

It's obvious, that the general solution of the mixed system of Cauchy type depends on no more then $\frac{1}{2}n(n^4 - 1)$ essential parameters.

References

[1] BEREZOVSKII, V., BÁCSÓ, S., MIKEŠ, J., Almost geodesic mappings of affinely connected spaces that preserve the Riemannian curvature, *Ann. Math. Inform.*, 45, 2015:3–10, ISSN 1787-5021.

[2] BEREZOVSKII, V., GUSEVA, N., MIKEŠ, J., On special first-type almost geodesic mappings of affine connection spaces preserving a certain tensor, *Math. Notes*, 98(3–4), 2015:515–518, ISSN 0001-4346, doi:10.1134/S0001.

[3] BEREZOVSKII, V., JUKL, M., JUKLOVÁ, L., Almost geodesic mappings of the first type onto symmetric spaces, in *6th Conf. on App. Math. APLIMAT 2017, Proc.*, 2017, ISBN 978-80-227-4765-3, pp. 126–131.

[4] BEREZOVSKII, V., MIKEŠ, J., On the classification of almost geodesic mappings of affine-connected spaces, in *Differential geometry and its applications (Dubrovnik, 1988)*, Univ. Novi Sad, Novi Sad, 1989, pp. 41–48.

[5] BEREZOVSKII, V., MIKEŠ, J., On a classification of almost geodesic mappings of affine connection spaces, *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.*, 35, 1996:21–24, ISSN 0231-9721.

[6] BEREZOVSKII, V., MIKEŠ, J., On almost geodesic mappings of the type π_1 of Riemannian spaces preserving a system n-orthogonal hypersurfaces, *Rend. Circ. Mat. Palermo (2) Suppl.*, (59), 1999:103–108, ISSN 1592-9531.

[7] BEREZOVSKII, V., MIKEŠ, J., Almost geodesic mappings of type π_1 onto generalized Ricci-symmetric manifolds, *Kazan. Gos. Univ. Uchen. Zap. Ser. Fiz.-Mat. Nauki*, 151(4), 2009:9–14.

[8] BEREZOVSKII, V., MIKEŠ, J., On canonical almost geodesic mappings of the first type of affinely connected spaces, *Russian Math. (Iz. VUZ)*, 58(2), 2014:1–5, ISSN 1066-369X, doi:10.3103/S1066369X14020017, translation of Izv. Vyssh. Uchebn. Zaved. Mat. 2014, no. 2, 3–8.

[9] BEREZOVSKII, V., MIKEŠ, J., Almost Geodesic mappings of spaces with affine connection, *J. Math. Sci.*, 207(3), 2015:389–409.

[10] BEREZOVSKII, V., MIKEŠ, J., CHUDÁ, H., CHEPURNAYA, E., Canonical almost geodesic mappings that preserve the projective curvature tensor, *Russian Math. (Iz. VUZ)*, (6), 2017:1–5, ISSN 1066-369X, doi:10.3103/s1066369x17060019.

[11] BEREZOVSKII, V., MIKEŠ, J., VANŽUROVÁ, A., Almost geodesic mappings onto generalized Ricci-symmetric manifolds, *Acta Math. Acad. Paedagog. Nyházi. (N.S.)*, 26(2), 2010:221–230, ISSN 1786-0091.

[12] BEREZOVSKII, V. E., MIKEŠ, J., VANŽUROVÁ, A., Fundamental PDE's of the canonical almost geodesic mappings of type $\tilde{\pi}_1$, *Bull. Malays. Math. Sci. Soc. (2)*, 37(3), 2014:647–659, ISSN 0126-6705.

[13] CARTAN, É., Les espaces riemanniens symétriques, *Verhandlungen Kongress Zürich*, 1, 1932:152–161.

[14] HELGASON, S., *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*, AMS, Providence, 2001, doi:10.1090/gsm/034.

[15] HINTERLEITNER, I., MIKEŠ, J., Geodesic mappings onto Weyl manifolds, in *APLIMAT 2009: 8th Int. Conf., Proc.*, Bratislava, 2009, pp. 423–430.

[16] LEVI-CIVITA, T., On the transformations of the dynamical equations, *Regul. Chaotic Dyn.*, 14(4-5), 2009:580–614, transl. from the Italian [Ann. Mat. **2**4 (1896), 255–300].

[17] MIKEŠ, J., Geodesic mappings of semisymmetric Riemannian spaces, *Odessk. Univ. Moscow: Archives at VINITI*, 3924-76(2), 1976:1–19.

[18] MIKEŠ, J., Geodesic mappings of Ricci 2-symmetric Riemannian spaces, *Mat. Notes*, 28(2), 1980:622–624, doi:10.1007/BF01157926.

[19] MIKEŠ, J., Geodesic mappings of affine-connected and Riemannian spaces, *J. Math. Sci.*, 78(3), 1996:311–333, ISSN 1072-3374, doi:10.1007/BF02365193.

[20] MIKEŠ, J., Holomorphically projective mappings and their generalizations, *J. Math. Sci. (New York)*, 89(3), 1998:1334–1353, ISSN 1072-3374, doi:10.1007/BF02414875.

[21] MIKEŠ, J., BEREZOVSKII, V. E., STEPANOVA, E., CHUDÁ, H., Geodesic mappings and their generalizations, *J. Math. Sci. (N.Y.)*, 217(5), 2016:607–623, ISSN 1072-3374, doi:10.1007/s10958-016-2993-9.

[22] MIKEŠ, J., RÝPAROVÁ, L., CHUDÁ, H., On theory of rotary mappings, *Math. Notes*, 104(3–4), 2018:617–620, ISSN 1573-8876, doi:10.1134/S0001434618090286.

[23] MIKEŠ, J., STEPANOVA, E., VANŽUROVÁ, A., ET AL., *Differential geometry of special mappings*, Palacky University, Olomouc, 2015, ISBN 978-80-244-4671-4, 568 pp.

[24] PETROV, A. Z., Simulation of physical fields, in *Gravitation and the Theory of Relativity, No. 4–5 (Russian)*, Izdat. Kazan. Univ., Kazan, 1968, pp. 7–21.

[25] SHIROKOV, A. P., P. A. Shirokov's work on the geometry of symmetric spaces, *J. Math. Sci. (New York)*, 89(3), 1998:1253–1260, doi:10.1007/BF02414870.

[26] SINYUKOV, N. S., Almost geodesic mappings of affinely connected and Riemannian spaces, *Dokl. Akad. Nauk SSSR*, 151, 1963:781–782, ISSN 0002-3264.

[27] SINYUKOV, N. S., *Geodesic mappings of Riemannian spaces*, Nauka, Moscow, 1979, 256 pp.

[28] SINYUKOV, N. S., Almost geodesic mappings of affinely connected and Riemannian spaces, in *Problems in geometry, Vol. 13*, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, Itogi Nauki i Tekhniki, 1982, pp. 3–26, 199.

[29] SOBCHUK, V. S., An almost geodesic mapping of Riemannian spaces onto symmetric Riemannian spaces, *Mat. Zametki*, 17(5), 1975:757–763, ISSN 0025-567X.

[30] SOBCHUK, V. S., Intrinsic almost geodesic mappings, *Izv. Vyssh. Uchebn. Zaved. Mat.*, (5), 1989:62–64, ISSN 0021-3446.

[31] VAVŘÍKOVÁ, K., MIKEŠ, U., POKORNÁ, O., STARKO, G., On the basic equations of the almost geodesic mappings $\pi_2(e)$, *Izv. Vyssh. Uchebn. Zaved. Mat.*, (1), 2007:10–15, ISSN 0021-3446, doi:10.3103/S1066369X07010021.

[32] VESIĆ, N. O., VELIMIROVIĆ, L. S., STANKOVIĆ, M. S., Some invariants of equitorsion third type almost geodesic mappings, *Mediterr. J. Math.*, 13(6), 2016:4581–4590, ISSN 1660-5446, doi:10.1007/s00009-016-0763-z.

[33] YABLONSKAYA, N. V., Special groups of almost geodesic transformations of spaces with affine connection, *Izv. Vyssh. Uchebn. Zaved. Mat.*, (1), 1986:78–80, 84, ISSN 0021-3446.

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