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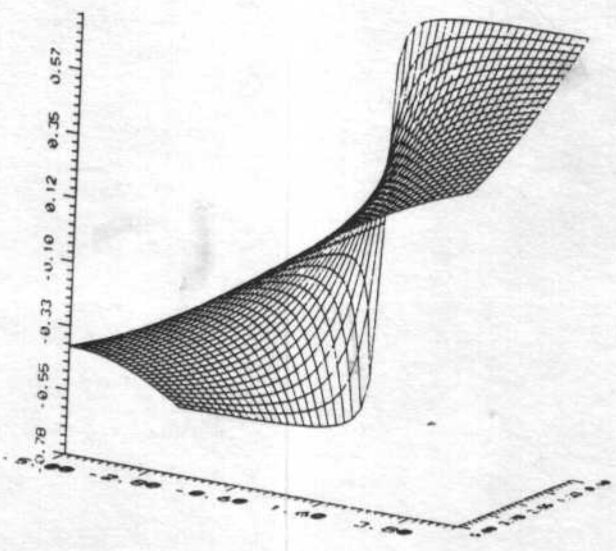

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ON THE CLASSIFICATION OF ALMOST GEODESIC MAPPINGS OF AFFINE-CONNECTED SPACES

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ABSTRACT

It is shown that except for π_1, π_2 and π_3 other almost geodesic mappings of affine-connected spaces (without the torsion and with it) do not exist if the dimension of space is $n > 5$. An analogous assertion is true for infinitesimal almost geodesic transformations.

The present paper is devoted to an investigation of the complete classification of almost geodesic mappings of affine-connected spaces without torsions. It is proved that almost geodesic mappings of spaces with affine connections A_n without torsions can only be of π_1, π_2 and π_3 types.

Let us recall some basic conceptions.

DEFINITION 1. [3]. *A curve of a space with affine connection* A_n *is called an almost geodesic line, if its tangential vector* $\lambda^h \overset{\text{def}}{=} dx^h/dt$ *satisfies the equations* $\lambda_2^h = a(t)\lambda^h + b(t)\lambda_1^h$ $\lambda_1^h \overset{\text{def}}{=} \lambda^h{}_{,\alpha}\lambda^\alpha$ $\lambda_2^h \overset{\text{def}}{=} \lambda_1^h{}_{,\alpha}\lambda^\alpha$ *were the comma denotes the covariant derivative with respect to the connection* A_n a *and* b *are*

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functions of a parameter t.

DEFINITION 2. [3]. *A mapping* f *of the space with affine connection* A_n *onto a space with affine connection* $\bar{A}_n$ *is called an almost geodesic mapping if any geodesic line of the space* A_n *turns into the almost geodesic line of the space* $\bar{A}_n$.

THEOREM: [3]. *In order that the mapping of* A_n *onto* $\bar{A}_n$ *should be almost geodesic, it is necessary and sufficient that in the general, respective to the mapping coordinate system, the deformation tensor* $P^h_{ij}(x)$ *should satisfy identically the conditions*

$$(1) \qquad A^h_{\alpha\beta\gamma}\,\lambda^\alpha\lambda^\beta\lambda^\gamma = a\,\lambda^h + b\,P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta\ ,$$

respective to $x^1, x^2, \dots, x^n$ *and* $\lambda^1, \lambda^2, \dots, \lambda^n$, *where the latter are components of an arbitrary vector;* a *and* b *are invariants, dependent on* $x^1, \dots, x^n, \lambda^1, \dots, \lambda^n$; $A^h_{\alpha\beta\gamma} \overset{\text{def}}{=} P^h_{(\alpha\beta,\gamma)} + P^h_{\varepsilon(\alpha}P^\varepsilon_{\beta\gamma)}$, *the brackets denote the symmetrization with the division,* $P^h_{ij}(x) = \bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x)$; $\Gamma^h_{ij}(\bar{\Gamma}^h_{ij})$ *are affine connections of* A_n $(\bar{A}_n)$.

According to the dependence of invariants a and b on $\lambda^1, \lambda^2, \dots, \lambda^n$, N.S.Sinyukov [3] singled out the tree types of the almost geodesic mappings, π_1, π_2, π_3. Namely,

(1) *The mapping is said to the almost geodesic of* π_1 *type, if*

$$(2) \qquad A^h_{(ijk)} = a_{(ij}\delta^h_{k)} + b_{(i}P^h_{jk)}\ ,$$

where a_{ij} *and* b_i *are tensors.*

(2) *The mapping is said to the almost geodesic of* π_2 *type, if*

$$(3a)\qquad P^h_{ij} = \delta^h_{(i}\psi_{j)} + F^h_{(i}\phi_{j)},$$

$$(3b)\qquad F^h_{(i,j)} = \delta^h_{(i}\eta_{j)} + F^h_{(i}\rho_{j)} + F^h_\alpha F^\alpha_{(i}\tau_{j)} ;$$

where F^h_i *is an affinor, and* ψ_i, ϕ_i, η_i, ρ_i, τ_i *are covectors.*

(3.) *The mapping is said to the almost geodesic of* π_3 *type, if*

$$(4a)\qquad P^h_{ij} = \delta^h_{(i}\psi_{j)} + \phi^h\,\omega_{ij},$$

$$(4b)\qquad \phi^h_{,i} = \phi^h\Theta_i + \rho\delta^h_i ,$$

where ϕ^h, ψ_i, Θ_i *are vectors,* ρ *is an invariant, and* ω_{ij} *is a symmetric tensor.*

One should note that these types of almost geodesic mappings can intersect.

It takes the following result.

THEOREM. *Only three types,* π_1, π_2 *and* π_3, *of almost geodesic mappings of spaces with affine connection* A_n *onto* $\bar{A}_n$ $(n>5)$ *can exist.*

P r o o f.

Let us consider an almost geodesic mapping of spaces with affine connection A_n onto $\bar{A}_n$. With these mappings, conditions of necessity and sufficiency (1) must be satisfied. Obviously, the basic equations of the almost geodesic mappings π_1, π_2 and π_3 are simpler than the basic equations (1). This is due to the fact that in (1), coordinates of an arbitrary vector λ^h were included as well as functions a and b, dependent on the coordinates of a point $(x^1, x^2,\dots,x^n)$ and vector λ^h. Statisfying one of the relations, either (2) or (3) or (4), leads to the realization of (1). The proof of the theorem obviously, is concerned with proving the inverse, namely, that (1) leads to the satisfying of either conditions (2) or (3) or (4).

Let us consider relation (1). Multiplying it by vectors λ^h and $P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta$ with the following alternation with respect to the indices h, i, j, we have

$$(5)\qquad A^{[h}_{\alpha\beta\gamma}\; P^{i}_{\delta\varepsilon}\; \delta^{j]}_{\eta}\; \lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta\lambda^\varepsilon\lambda^\eta = 0,$$

where $[i,j,k]$ denote the alternation.

This relation is a homogeneous six-order polynomial with respect to a component of an arbitrary vector λ^h and tensors $A^h_{\alpha\beta\gamma}$, $P^h_{\delta\varepsilon}$ are dependent only on coordinates of a point x^h. By virtue of the arbitrariness of λ^h, condition (5) is equal to the relation

$$(6)\qquad A^{[h}_{(\alpha\beta\gamma}\; P^{i}_{\delta\varepsilon}\; \delta^{j]}_{\eta)} = 0.$$

Conditions (5) and (6) are the basic equations of the almost geodesis mappings, as well. In particular, a mapping is said to be the almost geodesis if and only if the deformation tensor P^h_{ij} satisfies equation (6).

In the following we shall exclude from consideration the situation when the almost geodesis mapping is the geodesic one. The latter are the partial, but well studied case of almost geodesic mappings. Hence, we shall suppose that the deformation tensor P^h_{ij} satisties the condition

$$(7)\qquad P^h_{ij} \neq \delta^h_i\psi_j + \delta^h_j\psi_i,$$

where ψ_i is a covector.

Condition (7) provides the existence of such a vector ε^i that p^i and $p^h \equiv P^h_{\alpha\beta}\varepsilon^\alpha\varepsilon^\beta$ are not colinear. Then, contracting (6) by $\varepsilon^\alpha\varepsilon^\beta\varepsilon^\gamma\varepsilon^\delta\varepsilon^\varepsilon\varepsilon^\eta$ and by virtue of the noncolinearity of the vectors ε^i and p^i, we get

$$(8) \qquad A^h_{\alpha\beta\gamma}\,\varepsilon^\alpha\varepsilon^\beta\varepsilon^\gamma = \underset{1}{W}\,\varepsilon^h + \underset{2}{W}\,p^h,$$

where W_1 and W_2 are invariants. Contracting (6) by $\varepsilon^\gamma\varepsilon^\delta\varepsilon^\varepsilon\varepsilon^\eta\varepsilon^\beta$ $\varepsilon^\gamma\varepsilon^\delta\varepsilon^\varepsilon\varepsilon^\eta$ and $\varepsilon^\delta\varepsilon^\varepsilon\varepsilon^\eta$, in turns, and taking into account (8) and the intermediate results we have obtained, in total we are convinced that the tensor A^h_{ijk} can be represented as

$$(9) \qquad A^h_{ijk} = P^h_{(ij}b_{k)} + \delta^h_{(i}a_{jk)} + F^h_{(i}\overset{1}{W}_{jk)} + \varepsilon^h \overset{2}{W}_{ijk} + p^h\,\overset{3}{W}_{ijk},$$

where F^h_i, a_{jk}, $\overset{1}{W}_{jk}$, $\overset{2}{W}_{ijk}$, $\overset{3}{W}_{ijk}$, b_i are tensors.

Thus, the following assertion is proved.

LEMMA. *If A_n admits an almost geodesic mapping onto $\bar{A}_n$, different from the geodesic one, then condition (9) is satisfied.*

Obviously, if the almost geodesic mapping is different from mapping π_1, then the tensor

$$(10) \qquad \bar{A}^h_{ijk} \equiv F^h_{(i}\overset{1}{W}_{jk)} + \varepsilon^h \overset{2}{W}_{ijk} + p^h\,\overset{3}{W}_{ijk} \neq 0.$$

In the following we shall consider the almost geodesic mapping, different from π_1, for which (10) is true. Then, using (9) we shall exclude from (6) the tensor, and after transformations we get

$$(11) \qquad \bar{A}^{[h}_{(\alpha\beta\gamma}\,P^{i}_{\delta\varepsilon}{}^{j]}_{\eta)} = 0.$$

It can be shown that condition (10) provides the existence of vector $\bar{\varepsilon}^i$, such that

$$(\bar{A}^h_{\alpha\beta\nu}\,\delta^i_\varepsilon - \bar{A}^i_{\alpha\beta\nu}\,\delta^h_\varepsilon)\;\bar{\varepsilon}^\alpha\bar{\varepsilon}^\beta\bar{\varepsilon}^\nu\bar{\varepsilon}^\varepsilon \neq 0.$$

Using then, an analogous contraction of (11), by turn, we get

$$P^h_{ij} = \delta^h_{(i}\psi_{j)} + F^h_{(i}\phi_{j)} + \varepsilon^h\psi^1_{ij} + p^h\psi^2_{ij} + \bar{\varepsilon}^h\psi^3{}_{ij} + \bar{p}^h\psi^4{}_{ij}, \tag{12}$$

where ψ^1_{ij} , ψ^2_{ij}, ψ^3_{ij}, ψ^4_{ij}, $\bar{p}^h$ are tensors.

Let us denote

$$\bar{P}^h_{ij} = \varepsilon^h\psi^1{}_{ij} + p^h\psi^2{}_{ij} + \bar{\varepsilon}^h\psi^3{}_{ij} + \bar{p}^h\psi^4_{ij} \tag{13}$$

If $\bar{P}^h_{ij} = 0$, then from (12), there follows (3a). Naturally, we suppose that $\phi_i \neq 0$, then condition (1) can be written in the form

$$(F^h_{\alpha,\beta} + F^h_\varepsilon F^\varepsilon_\alpha \phi_\beta)\,\lambda^\alpha\lambda^\beta\phi_\nu\lambda^\nu = \tilde{a}\;\lambda^h + \tilde{b}\,F^h_\alpha\,\lambda^\alpha$$

where a and b are invariante dependent of x^h and λ^h. Since $\psi_i \neq 0$, it is not difficult to obtain from the latest

$$(F^h_{\alpha,\beta} + F^h_\varepsilon\,F^\varepsilon_\alpha\,\phi_\beta)\,\lambda^\alpha\lambda^\beta = \tilde{\tilde{a}}\,\lambda^h + \tilde{\tilde{b}}\,F^h_\alpha\,\lambda^\alpha \tag{14}$$

Assuming that $F^h_i \neq \rho\delta^h_i + \eta_i\,\Theta^h$ and using the methods described in [2], the correctness of (3b) follows from (14), i.e. on these conditions the mapping is on almost geodesic mapping of the π_2 type.

When $F^h_i = \rho\delta^h_i + \eta_i\,\Theta^h$, in this case the deformation tensor P^h_{ij} takse the form (4a). Substituting them into (1), we get

$$a_{\alpha\beta}\lambda^\alpha\lambda^\beta\Theta^h{}_{,\nu}\lambda^\nu = \tilde{a}\,\lambda^h + \tilde{b}\,\Theta^h\ ,$$

where $\tilde{a}$, $\tilde{b}$ are invariants dependent on x^h and λ^h.

Since $a_{ij} \neq 0$, it is easy to obtain from these relations

$$\Theta^h{}_{,\alpha}\lambda^\alpha = \tilde{\tilde{a}}\lambda^h + \tilde{\tilde{b}}\,\Theta^h.$$

From this relation it is easy to obtain (4b), i.e. the mapping is on almost geodesic of the π_3 type.

Assuming $\bar{P}^h_{ij} \neq 0$ we get

$$(15)\quad P^h_{ij} = \delta^h_{(i}\psi_{j)} + \Theta^h\omega_{ij} + \varepsilon^h\underset{1}{\omega}_{ij} + p^h\underset{2}{\omega}_{ij} + \varepsilon^h\underset{3}{\omega}_{ij} + p^h\underset{4}{\omega}_{ij}.$$

Using equations (1) and (15) and taking into account that the space dimension $n > 5$, it is easy to show that the deformation tensor P^h_{ij} takes either form (2) or (3a) or (4a). Thus, the theorem has been proved.

Analogous conceptions were introduced for almost geodesic mappings of affine-connected spaces with a torsion [5], and the theorem is true in this case too.

In [4] infinitesimal almost geodesic transformations are considered. It is true for them that other infinitesimal almost geodesic transformation, different from π_1, π_2 and π_3 types, do not exist.

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