mathematics

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Canonical Almost Geodesic Mappings of the First Type of Spaces with Affine Connections onto Generalized *m*-Ricci-Symmetric Spaces

Volodymyr Berezovski [1], Yevhen Cherevko [2,3], Josef Mikeš [4,*] and Lenka Rýparová [5]

[1] Department of Mathematics and Physics, Uman National University of Horticulture, 20300 Uman, Ukraine; berez.volod@rambler.ru
[2] Department of Physics and Mathematics Sciences, Odesa National Academy of Food Technologies, 65039 Odesa, Ukraine; cherevko@usa.com
[3] Department of Economic Cybernetics and Information Technologies, Odesa National Economics University, 65082 Odesa, Ukraine
[4] Department of Algebra and Geometry, Faculty of Science, Palacky University, 771 46 Olomouc, Czech Republic
[5] Institute of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Brno University of Technology, 602 00 Brno, Czech Republic; ryparova.l@fce.vutbr.cz
* Correspondence: josef.mikes@upol.cz

Abstract: In the paper we consider almost geodesic mappings of the first type of spaces with affine connections onto generalized 2-Ricci-symmetric spaces, generalized 3-Ricci-symmetric spaces, and generalized m-Ricci-symmetric spaces. In either case the main equations for the mappings are obtained as a closed system of linear differential equations of Cauchy type in the covariant derivatives. The obtained results extend an amount of research produced by N.S. Sinyukov, V.E. Berezovski, J. Mikeš.

Keywords: canonical almost geodesic mappings; Cauchy-type PDEs; space with affine connection; Ricci symmetric space

MSC: 53B05, 35R01

Citation: Berezovski, V.; Cherevko, Y.; Mikeš, J.; Rýparová, L. Canonical Almost Geodesic Mappings of the First Type of Spaces with Affine Connections onto Generalized *m*-Ricci-Symmetric Spaces. *Mathematics* **2021**, *9*, 437. https://doi.org/10.3390/math9040437

Academic Editor: Marian Ioan Munteanu

Received: 26 January 2021
Accepted: 18 February 2021
Published: 22 February 2021

1. Introduction

In the sixties of the preceding century, N.S. Sinyukov [1] considered almost geodesic mappings of Riemannian and affinely connected spaces, that generalize the geodesic mappings in a natural way. The main results were presented in a monograph [2] and an expository article [3]. Geodesic mappings and their generalizations are studied in detail in monographs [4–6] and researches [7–9] by J. Mikeš et al.

The theory goes back to the paper [10] by T. Levi-Civita, in which the problem on the search for Riemannian spaces with common geodesics was stated and solved in a special coordinate system. We note a remarkable fact that this problem is related to the study of equations of dynamics of mechanical systems. That direction is developing according to Petrov's plan to build models of physical processes using mappings and transformations [11]. For example, in papers [12,13] is shown the possible physical application of almost geodesic mappings.

N.S. Sinyukov specified three types of almost geodesic mappings π_1, π_2, π_3. The problem of completeness of classification had long remained unresolved. Berezovski and Mikeš [14,15] proved that for $n > 5$ other types of almost geodesic mappings except for π_1, π_2, and π_3 do not exist. The authors have found conditions for the almost geodesic mappings π_1, π_2, π_3 intersections. It is proved that if an almost geodesic mapping f is simultaneously π_1 and π_2, then f is a mapping of affine connection spaces with a preserved linear complex of geodesic lines. If the mapping f is simultaneously π_1 and π_3, then f is a mapping of affine connection spaces with a preserved quadratic complex of geodesic lines. The mappings preserving above mentioned complexes studied by V.M. Chernyshenko [16] and V.A. Dobrovolski [17].

The theory of almost geodesic mappings was developed by V.A. Aminova, A.M. Mukhamedov [18], V.S. Sobchuk [19,20], N.Y. Yablonskaya [21,22], V.E. Berezovski, J. Mikeš [14,15,23–35], O. Belova, J. Mikeš, K. Strambach [36,37], M.S. Stankovič, Lj.S. Velimirović, N. Vesić, M.Lj. Zlatanović [38–44] et al.

N.S. Sinyukov [3] proved that the main equations for canonical almost geodesic mappings of spaces with affine connections onto Ricci-symmetric spaces can be written as a closed system of partial differential equations of Cauchy type in covariant derivatives. It follows that the general solution of the system depends on a finite number of essential parameters.

The results were extended by Berezovski and Mikeš [28,34] to the cases of canonical almost geodesic mappings of the first type of spaces with affine connections onto Riemannian spaces and canonical almost geodesic mappings of the first type of spaces with affine connections onto generalized Ricci-symmetric spaces.

The paper is devoted to the study of canonical almost geodesic mappings of type π_1 of spaces with affine connections onto generalized 2-Ricci-symmetric, generalized 3-Ricci-symmetric, and generalized m-Ricci-symmetric spaces. The fundamental equations for the mappings are derived in the form of the closed systems of Cauchy-type PDEs. The maximum number of essential parameters on which general solutions of these systems depend was estimated. V.R. Kaigorodov [45] comprehensively studied generalized symmetric and recurrent spaces from the point of view of the General Theory of Relativity. This review contains 97 citations and is a through analysis of this issue.

Let us note that in works [7,8,24,46–50] were studied geodesic and holomorphically projective mappings of above mentioned spaces. For the other mappings the main equations are obtained as closed systems of PDEs of Cauchy type, see [5,6,51–54].

2. Basic Definitions of Almost Geodesic Mappings of Spaces with Affine Connections

Let us recall the basic definition, formulas and theorems of the theory presented in [2,5,6].

Consider an n-dimensional space A_n with affine connection Γ_{ij}^h without torsion. The space is referred to coordinates $x^1, x^2, \dots, x^n$. We assume that all functions under consideration are sufficiently differentiable, and we suppose that $n > 2$.

A curve defined in a space with affine connection is called *almost geodesic* if there exists a two-dimensional (differentiable) distribution D parallel along the curve (relative to the affine connection) such that for any tangent vector of the curve its parallel translation along the curve belongs to the distribution D [5].

A mapping $f\colon A_n \to \overline{A}_n$ is called *almost geodesic* if any geodesic curve of A_n is mapped under f onto an almost geodesic curve in $\overline{A}_n$.

Suppose, that a space A_n with affine connection $\Gamma_{ij}^h(x)$ admits a mapping f onto a space $\overline{A}_n$ with affine connection $\overline{\Gamma}_{ij}^h(x)$, and the spaces are referred to a common coordinate system $x^1, x^2, \dots, x^n$ with respect to the mapping.

The tensor

$$P_{ij}^h(x) = \overline{\Gamma}_{ij}^h(x) - \Gamma_{ij}^h(x) \quad (1)$$

is called a *deformation tensor* of the connections $\Gamma_{ij}^h(x)$ and $\overline{\Gamma}_{ij}^h(x)$ with respect to the mapping f. The symbols $\Gamma_{ij}^h(x)$ and $\overline{\Gamma}_{ij}^h(x)$ are components of affine connections of the spaces A_n and $\overline{A}_n$ respectively.

It is known [2] that in order that a mapping of a space A_n onto a space $\overline{A}_n$ to be almost geodesic, it is necessary and sufficient that the deformation tensor $P^h_{ij}(x)$ of the mapping f in the common coordinate system $x^1, x^2, \ldots, x^n$ has to satisfy the condition

$$A^h_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma = a \cdot P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + b \cdot \lambda^h,$$

where λ^h is an arbitrary vector, a and b are certain functions of variables $x^1, x^2, \ldots, x^n$ and $\lambda^1, \lambda^2, \ldots, \lambda^n$. The tensor A^h_{ijk} is defined as

$$A^h_{ijk} \stackrel{\text{def}}{=} P^h_{ij,k} + P^\alpha_{ij}P^h_{\alpha k},$$

where the comma "," denotes the covariant derivative with respect to the connection of the space A_n.

N. S. Sinyukov [2] distinguished three kinds of almost geodesic mappings, namely π_1, π_2, and π_3 characterized, by following conditions for the deformation tensor.

A mapping $f : A_n \to \overline{A}_n$ is called an almost geodesic of type π_1, if the conditions

$$A^h_{(ijk)} = \delta^h_{(i}a_{jk)} + b_{(i}P^h_{jk)}, \tag{2}$$

are satisfied, where a_{ij} is a symmetric tensor, b_i is a covariant vector and δ^h_i is the Kronecker delta. We denote by the round parentheses a *symmetrization* without division with respect to the mentioned indices.

If in the Equation (2) the condition $b_i \equiv 0$ holds, then the almost geodesic mapping π_1 is called *canonical*. It is known [3] that any almost geodesic mapping π_1 may be written as the composition of a canonical almost geodesic mapping of type π_1 and a geodesic mapping.

A mapping $f : A_n \to \overline{A}_n$ is called an almost geodesic of type π_2, if the conditions

$$P^h_{ij} = \delta^h_{(i}\psi_{j)} + F^h_{(i}\varphi_{j)} \quad \text{and} \quad F^h_{(i,j)} + F^h_\alpha F^\alpha_{(i}\varphi_{j)} = \delta^h_{(i}\mu_{j)} + F^h_{(i}\rho_{j)}$$

holds, where ψ_i, φ_i, μ_i, ρ_i are covectors, F^h_i is a tensor of type $(1,1)$.

A mapping $f : A_n \to \overline{A}_n$ is called an almost geodesic of type π_3, if the conditions

$$P^h_{ij} = \delta^h_{(i}\psi_{j)} + \theta^h a_{ij} \quad \text{and} \quad \theta^h_{,i} = \rho \cdot \delta^h_i + \theta^h a_i$$

holds, where θ^h is a torse forming vector, ψ_i, a_i are covectors, a_{ij} is a symmetric tensor and ρ is a function.

As we have noted above, the mentioned types of mappings π_1, π_2 and π_3 can intersect. The classification completeness for spaces with affine connections of dimension $n > 5$ has been proved in [14,15].

3. Ricci-Symmetric, Generalized Ricci-Symmetric, 2-Ricci-Symmetric and Generalized 2-Ricci-Symmetric Spaces

A space $\overline{A}_n$ with an affine connection is called *Ricci-symmetric* if its Ricci tensor $\overline{R}_{ij}$ satisfies the condition

$$\overline{R}_{ij;k} = 0.$$

The symbol ";" denotes a covariant derivative with respect to the connection of the space $\overline{A}_n$.

In [3] N.S. Sinyukov considered canonical almost geodesic mappings of the first type of spaces with affine connections onto Riemannian Ricci-symmetric spaces. Taking into account the relations between the Riemannian tensors $\overline{R}^h_{ijk}$ and R^h_{ijk} of the spaces $\overline{A}_n$ and A_n

respectively as follows from (1), the Equations (2) for canonical almost geodesic mappings of the first type of spaces with affine connections were written in the form

$$3\left(P^h_{ij,k} + P^h_{k\alpha}P^\alpha_{ij}\right) = R^h_{(ij)k} - \overline{R}^h_{(ij)k} + \delta^h_{(k}a_{ij)}. \tag{3}$$

We give the following example. From the Equation (3) with a condition of preserving Riemannian tensor and $a_{ij} = 0$, we obtain

$$P^h_{ij,k} = -P^h_{k\alpha}P^\alpha_{ij}.$$

In a flat space, the above equation is completely integrable. Therefore it has a solution with respect to $P^h_{ij}(x)$ for any initial condition $P^h_{ij}(x_0)$. If $P^hij(x_0) \neq \delta^h_{(i}\psi_{j)}(x_0)$ then the solution $P^h_{ij}(x_0)$ generates an almost geodesic mapping $A_n \to \bar{A}_n$ which is not geodesic.

Considering (3) as a system of Cauchy type in A_n with respect to the functions P^h_{ij}, one has found the integrability conditions of the system in the form

$$\begin{aligned}\overline{R}^h_{(ij)[k,l]} &= R^h_{(ij)[k,l]} + \delta^h_{(i}a_{jk),l} - \delta^h_{(i}a_{jl),k} + 3\left(P^\alpha_{ij}\overline{R}^h_{\alpha kl} - P^h_{\alpha(j}R^\alpha_{i)kl}\right) \\ &\quad - P^h_{\alpha k}\left(R^h_{(ij)l} - \overline{R}^h_{(ij)l} + \delta^\alpha_{(i}a_{jl)}\right) + P^h_{\alpha l}\left(R^h_{(ij)k} - \overline{R}^h_{(ij)k} + \delta^\alpha_{(i}a_{jk)}\right),\end{aligned}$$

where the square brackets [ ] denote an *antisymmetrization* (or, *alternation*) without division with respect to the mentioned indices.

Let us express in the left-hand side the covariant derivatives with respect to the connection of A_n in terms of the covariant derivatives with respect to the connection of $\overline{A}_n$. Taking account of (1), we obtain

$$\overline{R}^h_{(ij)[k;l]} = \delta^h_{(i}a_{jk),l} - \delta^h_{(i}a_{jl),k} + \theta^h_{ijkl}, \tag{4}$$

where

$$\begin{aligned}\theta^h_{ijkl} &= R^h_{(ij)[k,l]} + 3\left(P^\alpha_{ij}\overline{R}^h_{\alpha kl} - P^h_{\alpha(j}R^\alpha_{i)kl}\right) - P^h_{\alpha k}\left(R^h_{(ij)l} - \overline{R}^h_{(ij)l} + \delta^\alpha_{(i}a_{jl)}\right) \\ &\quad + P^h_{\alpha l}\left(R^h_{(ij)k} - \overline{R}^h_{(ij)k} + \delta^\alpha_{(i}a_{jk)}\right) - P^\alpha_{l(i}\overline{R}^h_{|\alpha|j)k} - P^\alpha_{l(i}\overline{R}^h_{j)\alpha k} + P^\alpha_{k(i}\overline{R}^h_{|\alpha|j)l} + P^\alpha_{k(i}\overline{R}^h_{j)\alpha l}.\end{aligned}$$

Here the symbol $|\alpha|$ means that in applying symmetrization the index α is omitted.

Using the Ricci identity, let us write the conditions (4) in the form

$$\overline{R}^h_{ilk;j} + \overline{R}^h_{jlk;i} = \delta^h_{(i}a_{jk),l} - \delta^h_{(i}a_{jl),k} + \theta^h_{ijkl}. \tag{5}$$

Contracting (5) for h and k, we get the relations for covariant derivatives of the Ricci tensor $\overline{R}_{ij}$ for the space $\overline{A}_n$

$$\overline{R}_{il;j} + \overline{R}_{jl;i} = (n+1)a_{ij,l} - a_{l(i,j)} + \theta^\alpha_{ij\alpha l}. \tag{6}$$

A space $\overline{A}_n$ with an affine connection is called *generalized Ricci-symmetric* if the Ricci tensor $\overline{R}_{ij}$ for the space satisfies the condition

$$\overline{R}_{ij;k} + \overline{R}_{kj;i} = 0.$$

The concept of generalized Ricci-symmetric spaces was first introduced in [28,34]. The papers are devoted to the study of canonical almost geodesic mappings of spaces with affine connections onto the above mentioned spaces. It is proved that the main equations for the mappings can be obtained as a closed system of Cauchy-type differential equations in covariant derivatives.

A space $\overline{A}_n$ with an affine connection is called *2-Ricci-symmetric* if the Ricci tensor $\overline{R}_{ij}$ for the space satisfies the condition

$$\overline{R}_{ij;km} = 0.$$

A space $\overline{A}_n$ with an affine connection is called *generalized 2-Ricci-symmetric* if its Ricci tensor $\overline{R}_{ij}$ satisfies the condition

$$\overline{R}_{ij;km} + \overline{R}_{kj;im} = 0. \tag{7}$$

4. Canonical Almost Geodesic Mappings of type π_1 of Spaces with Affine Connections onto Generalized 2-Ricci-Symmetric Spaces

The Equation (6) was obtained for a canonical almost geodesic mapping of type π_1 of a space with an affine connection A_n onto another space with an affine connection $\overline{A}_n$.

First we differentiate covariantly the Equation (6) with respect to the connection of the space A_n. Then taking account of (1), let us applicate the formulas

$$\begin{aligned} \left(\overline{R}_{il;j}\right)_{,k} &= \overline{R}_{il;jk} + \overline{R}_{\alpha l;j} P^{\alpha}_{ik} + \overline{R}_{i\alpha;j} P^{\alpha}_{lk} + \overline{R}_{il;\alpha} P^{\alpha}_{jk}, \\ \overline{R}_{ij;k} &= \overline{R}_{ij,k} - \overline{R}_{i\alpha} P^{\alpha}_{jk} - \overline{R}_{\alpha j} P^{\alpha}_{ik}. \end{aligned} \tag{8}$$

Finally, we get

$$\overline{R}_{il;jk} + \overline{R}_{jl;ik} = (n+1) a_{ij,lk} - a_{li,jk} - a_{lj,ik} + C_{ijlk}, \tag{9}$$

where

$$\begin{aligned} C_{ijlk} = -\theta^{\alpha}_{ij\alpha l,k} &- \left(\overline{R}_{\alpha l,j} + \overline{R}_{jl,\alpha} - 2\overline{R}_{\beta l} P^{\beta}_{\alpha j} - \overline{R}_{\alpha\beta} P^{\beta}_{lj} - \overline{R}_{j\beta} P^{\beta}_{l\alpha}\right) P^{\alpha}_{ik} \\ &- \left(\overline{R}_{i\alpha,j} + \overline{R}_{j\alpha,i} - 2\overline{R}_{\beta\alpha} P^{\beta}_{ij} - \overline{R}_{i\beta} P^{\beta}_{\alpha j} - \overline{R}_{j\beta} P^{\beta}_{\alpha i}\right) P^{\alpha}_{lk} \\ &- \left(\overline{R}_{il,\alpha} + \overline{R}_{\alpha l,i} - 2\overline{R}_{\beta l} P^{\beta}_{\alpha i} - \overline{R}_{i\beta} P^{\beta}_{l\alpha} - \overline{R}_{\alpha\beta} P^{\beta}_{li}\right) P^{\alpha}_{jk}. \end{aligned} \tag{10}$$

Moreover, let us consider canonical almost geodesic mappings of type π_1 of spaces A_n with affine connections onto generalized 2-Ricci-symmetric spaces $\overline{A}_n$. Hence the Ricci tensor $\overline{R}_{ij}$ for the space $\overline{A}_n$ satisfies the conditions. Then the Equation (9) may be written in the form

$$(n+1) a_{ij,lk} - a_{li,jk} - a_{lj,ik} = -C_{ijlk}, \tag{11}$$

when C_{ijlk} is defined by the formulas (10).

Let us interchange the indices l and j in (11) and symmetrize in the indices i and j. Then we have

$$a_{li,jk} + a_{lj,ik} = -\frac{1}{n} C_{(i|l|j)k} + \frac{2}{n} a_{ij,lk}. \tag{12}$$

The Equation (11) by means of (12) can be written in the form

$$\frac{n^2+n-2}{n} a_{ij,lk} = -C_{ijlk} - \frac{1}{n} C_{(i|l|j)k}. \tag{13}$$

In the following we have assumed that a space A_n with an affine connection is given. Then taking account of the structure of the tensor C_{ijlk} which was determined by formula (10), we see that the left hand side of the Equation (13) depends on the unknown functions $P^h_{ij}(x)$, $a_{ij}(x)$, $a_{ij,k}(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijk,m}(x)$.

Differentiate covariantly the conditions of integrability (4) with respect to the connection of the space $\overline{A}_n$. Then express in the right-hand side the covariant derivatives with respect to the connection of $\overline{A}_n$ in terms of the covariant derivatives with respect to the connection of the space A_n. When we make use of the Ricci identity, we obtain

$$\overline{R}^h_{(ij)k;lm} - \overline{R}^h_{(ij)l;mk} = \delta^h_{(i} a_{jk),lm} - \delta^h_{(i} a_{jl),km} + T^h_{ijklm}, \tag{14}$$

where

$$T^h_{ijklm} = \overline{R}^h_{\alpha mk}\overline{R}^\alpha_{(ij)l} - \overline{R}^\alpha_{lmk}\overline{R}^h_{(ij)\alpha} - \overline{R}^\alpha_{jmk}\overline{R}^h_{(i\alpha)l} - \overline{R}^\alpha_{imk}\overline{R}^h_{(j\alpha)l} - P^h_{m\alpha}\delta^\alpha_{(i}a_{jk),l}$$
$$-P^\alpha_{mj}\delta^h_{(i}a_{\alpha k),l} - P^\alpha_{mi}\delta^h_{(\alpha}a_{jk),l} - P^\alpha_{mk}\delta^h_{(\alpha}a_{ij),l} - P^\alpha_{ml}\delta^h_{(i}a_{jk),\alpha} - P^h_{m\alpha}\delta^\alpha_{(i}a_{jl),k}$$
$$+P^\alpha_{mi}\delta^h_{(\alpha}a_{jl),k} + P^\alpha_{mj}\delta^h_{(i}a_{\alpha l),k} + P^\alpha_{mk}\delta^h_{(i}a_{jl),\alpha} - P^\alpha_{ml}\delta^h_{(i}a_{j\alpha),k}$$
$$-\theta^h_{ijkl,m} + P^h_{\alpha m}\theta^\alpha_{ijkl} - P^\alpha_{mi}\theta^h_{\alpha jkl} - P^\alpha_{mj}\theta^h_{i\alpha kl} - P^\alpha_{mk}\theta^h_{ij\alpha l} - P^\alpha_{ml}\theta^h_{ijk\alpha}.$$

Alternating the equations (14) with respect to the indicies l and m, we obtain

$$\overline{R}^h_{(ij)m;lk} - \overline{R}^h_{(ij)l;mk} = \delta^h_{(i}a_{jm),kl} - \delta^h_{(i}a_{jl),km} + T^h_{ijk[lm]} + \overline{R}^h_{(i|\alpha k|}\overline{R}^\alpha_{j)ml} + \overline{R}^h_{(ij)\alpha}\overline{R}^\alpha_{kml}$$
$$-\overline{R}^\alpha_{(ij)k}\overline{R}^h_{\alpha ml} + \overline{R}^h_{\alpha(i|k|}\overline{R}^\alpha_{j)ml} + \delta^h_{(\alpha}a_{jk)}R^\alpha_{ilm} + \delta^h_{(\alpha}a_{ik)}R^\alpha_{jlm} + \delta^h_{(i}a_{j\alpha)}R^\alpha_{klm} - \delta^\alpha_{(i}a_{jk)}R^h_{\alpha lm}. \quad (15)$$

Taking account of the properties of a curvature tensor R^h_{ijk}, we may write the conditions (15) in the form

$$\overline{R}^h_{iml;jk} + \overline{R}^h_{jml;ik} = \delta^h_{(i}a_{jl),km} - \delta^h_{(i}a_{jm),kl} - N^h_{ijklm}, \quad (16)$$

where

$$N^h_{ijklm} = T^h_{ijk[lm]} + \overline{R}^\alpha_{iml}\overline{R}^h_{(\alpha j)k} + \overline{R}^\alpha_{jml}\overline{R}^h_{(\alpha i)k} + \overline{R}^\alpha_{kml}\overline{R}^h_{(ij)\alpha} - \overline{R}^h_{\alpha ml}\overline{R}^\alpha_{(ij)k}$$
$$+\delta^h_{(\alpha}a_{jk)}R^\alpha_{ilm} + \delta^h_{(\alpha}a_{ik)}R^\alpha_{jlm} + \delta^h_{(\alpha}a_{ji)}R^\alpha_{klm} - a_{(ij}R^h_{k)lm}.$$

Alternating the equations (16) with respect to the indicies j and k, we get

$$\overline{R}^h_{jml;ik} - \overline{R}^h_{kml;ij} = \delta^h_{(i}a_{jl),km} - \delta^h_{(i}a_{jm),kl} - \delta^h_{(i}a_{kl),jm} + \delta^h_{(i}a_{km),jl}$$
$$-N^h_{i[jk]lm} + \overline{R}^h_{\alpha ml}\overline{R}^\alpha_{ikj} + \overline{R}^h_{i\alpha l}\overline{R}^\alpha_{mkj} + \overline{R}^h_{im\alpha}\overline{R}^\alpha_{lkj} - \overline{R}^\alpha_{iml}\overline{R}^h_{\alpha kj}. \quad (17)$$

Let us interchange i and k in (16) and subtract it from (17). Then we have

$$2\overline{R}^h_{jml;ik} = \delta^h_{(i}a_{jl),km} - \delta^h_{(i}a_{jm),kl} + \delta^h_{(k}a_{jm),il} + \delta^h_{(i}a_{km),jl} - \delta^h_{(i}a_{kl),jm} - \delta^h_{(k}a_{jl),im} + \Omega^h_{ijklm}, \quad (18)$$

where

$$\Omega^h_{ijklm} = N^h_{kjilm} - N^h_{i[jk]lm} + \overline{R}^h_{\alpha ml}\overline{R}^\alpha_{ikj} + \overline{R}^h_{i\alpha l}\overline{R}^\alpha_{mkj} + \overline{R}^h_{im\alpha}\overline{R}^\alpha_{lkj} - \overline{R}^\alpha_{iml}\overline{R}^h_{\alpha kj} - \overline{R}^\alpha_{kml}\overline{R}^h_{\alpha ji}$$
$$+\overline{R}^h_{\alpha ml}\overline{R}^\alpha_{kji} + \overline{R}^h_{k\alpha l}\overline{R}^\alpha_{mji} + \overline{R}^h_{km\alpha}\overline{R}^\alpha_{lji} - \overline{R}^\alpha_{jml}\overline{R}^h_{\alpha ik} + \overline{R}^h_{\alpha ml}\overline{R}^\alpha_{jik} + \overline{R}^h_{j\alpha l}\overline{R}^\alpha_{mik} + \overline{R}^h_{jm\alpha}\overline{R}^\alpha_{lik}.$$

Let us express in the left-hand side of the Equation (18) the covariant derivatives with respect to the connection of $\overline{A}_n$ in terms of the covariant derivatives with respect to the connection of A_n. We have

$$2\overline{R}^h_{jml,ik} = \delta^h_{(i}a_{jl),km} - \delta^h_{(i}a_{jm),kl} + \delta^h_{(k}a_{jm),il} + \delta^h_{(i}a_{km),jl} - \delta^h_{(i}a_{kl),jm} - \delta^h_{(k}a_{jl),im} + S^h_{ijklm}, \quad (19)$$

where

$$\begin{aligned}S^h_{ijklm} = \Omega^h_{ijklm} - 2\big[&\overline{R}^\alpha_{jml,i}P^h_{\alpha k} - \overline{R}^h_{\alpha ml,i}P^\alpha_{jk} - \overline{R}^h_{j\alpha l,i}P^\alpha_{mk} - \overline{R}^h_{jm\alpha,i}P^\alpha_{lk} - \overline{R}^h_{jml,\alpha}P^\alpha_{ik}\\
&+\big(\overline{R}^\alpha_{jml}P^\beta_{\alpha i} - \overline{R}^\beta_{\alpha ml}P^\alpha_{ij} - \overline{R}^\beta_{j\alpha l}P^\alpha_{im} - \overline{R}^\beta_{jm\alpha}P^\alpha_{il}\big)P^h_{\beta k}\\
&-\big(\overline{R}^\alpha_{jml}P^h_{\alpha\beta} - \overline{R}^h_{\alpha ml}P^\alpha_{\beta j} - \overline{R}^h_{j\alpha l}P^\alpha_{\beta m} - \overline{R}^h_{jm\alpha}P^\alpha_{\beta l}\big)P^\beta_{ik}\\
&-\big(\overline{R}^\alpha_{\beta ml}P^h_{\alpha i} - \overline{R}^h_{\alpha ml}P^\alpha_{\beta i} - \overline{R}^h_{\beta\alpha l}P^\alpha_{im} - \overline{R}^h_{\beta m\alpha}P^\alpha_{il}\big)P^\beta_{jk}\\
&-\big(\overline{R}^\alpha_{j\beta l}P^h_{\alpha i} - \overline{R}^h_{\alpha\beta l}P^\alpha_{ji} - \overline{R}^h_{j\alpha l}P^\alpha_{\beta i} - \overline{R}^h_{j\beta\alpha}P^\alpha_{il}\big)P^\beta_{km}\\
&-\big(\overline{R}^\alpha_{jm\beta}P^h_{\alpha i} - \overline{R}^h_{\alpha m\beta}P^\alpha_{ji} - \overline{R}^h_{j\alpha\beta}P^\alpha_{mi} - \overline{R}^h_{jm\alpha}P^\alpha_{\beta i}\big)P^\beta_{kl}\big].\end{aligned}$$

We introduce the tensors a_{ijk} and $\overline{R}^h_{ijkm}$ defined by

$$a_{ij,k} = a_{ijk}, \tag{20}$$

$$\overline{R}^h_{ijk,m} = \overline{R}^h_{ijkm}. \tag{21}$$

Taking account of (21), we may write the Equation (19) in the form

$$2\overline{R}^h_{jmli,k} = \delta^h_{(i}a_{jl),km} - \delta^h_{(i}a_{jm),kl} + \delta^h_{(k}a_{jm),il} + \delta^h_{(i}a_{km),jl} - \delta^h_{(i}a_{kl),jm} - \delta^h_{(k}a_{jl),im} + S^h_{ijklm}. \tag{22}$$

In the following we have assumed that in the left-hand side of the Equation (22) the second order covariant derivatives of the tensor a_{ij} are expressed according to (13). Taking account of (20), the Equation (13) may be put in the form

$$\frac{n^2+n-2}{n}\,a_{ijl,k} = -C_{ijlk} - \frac{1}{n}\,C_{(i|l|j)k}. \tag{23}$$

We have assumed that in the right-hand sides of the Equations (22) and (23) the covariant derivatives of tensors a_{ij} and $\overline{R}^h_{ijk}$ with respect to the connection of the space A_n are expressed according to (20) and (21).

Obviously, in the space A_n the Equations (3) and (20)–(23) form a closed system of PDEs of Cauchy type with respect to the functions $P^h_{ij}(x)$, $a_{ij}(x)$, $\overline{R}^h_{ijk}(x)$, $a_{ijk}(x)$, $\overline{R}^h_{ijkm}(x)$. The functions must satisfy the algebraic conditions

$$\begin{aligned}P^h_{ij}(x) = P^h_{ji}(x), \qquad & a_{ij}(x) = a_{ji}(x),\\ \overline{R}^h_{i(jk)}(x) = \overline{R}^h_{(ijk)}(x) = 0, \qquad & \overline{R}^h_{i(jk)l}(x) = \overline{R}^h_{(ijk)l}(x) = 0.\end{aligned} \tag{24}$$

Hence we have proved the following theorem.

Theorem 1. *In order that a space A_n with an affine connection admits a canonical almost geodesic mapping of type π_1 onto a generalized 2-Ricci-symmetric space $\overline{A}_n$, it is necessary and sufficient that the mixed system of differential equations of Cauchy type in covariant derivatives (3) and (20)–(24) have a solution with respect to the unknown functions $P^h_{ij}(x)$, $a_{ij}(x)$, $a_{ijk}(x)$, $\overline{R}^h_{ijk}(x)$ and $\overline{R}^h_{ijkm}(x)$.*

Also we have obtained the corollary.

Corollary 1. *The family of all generalized 2-Ricci-symmetric spaces, which are images of a given space A_n with an affine connection with respect to canonical almost geodesic mappings of type π_1, depends on no more then*

$$\frac{1}{6}\,n(2n-1)(n^2-1) + \frac{1}{2}\,n(1+n)^2$$

essential parameters.

5. Canonical Almost Geodesic Mappings of Type π_1 of Spaces with Affine Connections onto Generalized 3-Ricci-Symmetric Spaces

A space $\overline{A}_n$ with an affine connection is called *generalized 3-Ricci-symmetric* if the Ricci tensor $\overline{R}_{ij}$ for the space satisfies the condition

$$\overline{R}_{ij;kml} + \overline{R}_{kj;iml} = 0. \tag{25}$$

Taking account of (1) and (8), we obtain

$$\overline{R}_{ij;kl} = G_{ijkl} \quad \text{and} \quad \left(\overline{R}_{il;jk}\right)_{,m} = \overline{R}_{il;jkm} + G_{\alpha ljk}P^{\alpha}_{im} + G_{i\alpha jk}P^{\alpha}_{lm} + G_{il\alpha k}P^{\alpha}_{jm} + G_{ilj\alpha}P^{\alpha}_{km}, \tag{26}$$

where

$$G_{ijkl} = \overline{R}_{ij,kl} - \overline{R}_{i\alpha,l}P^{\alpha}_{jk} - \overline{R}_{i\alpha}P^{\alpha}_{jk,i} - \overline{R}_{\alpha j,l}P^{\alpha}_{ik} - \overline{R}_{\alpha j}P^{\alpha}_{ik,l} - \left(\overline{R}_{\alpha j,k} - \overline{R}_{\beta j}P^{\beta}_{\alpha k} - \overline{R}_{\alpha\beta}P^{\beta}_{jk}\right)P^{\alpha}_{il}$$
$$- \left(\overline{R}_{i\alpha,k} - \overline{R}_{\beta\alpha}P^{\beta}_{ik} - \overline{R}_{i\beta}P^{\beta}_{\alpha k}\right)P^{\alpha}_{jl} - \left(\overline{R}_{ij,\alpha} - \overline{R}_{\beta j}P^{\beta}_{i\alpha} - \overline{R}_{i\beta}P^{\beta}_{j\alpha}\right)P^{\alpha}_{kl}.$$

Let us differentiate covariantly the Equation (9) with respect to the connection of the space A_n. Then taking account of (8) and (26), we get

$$\overline{R}_{il;jkm} + \overline{R}_{jl;ikm} = (n+1)a_{ij,lkm} - a_{li,jkm} - a_{lj,ikm}$$
$$+ C_{ijlk,m} - G_{\alpha l(j|k|}P^{\alpha}_{i)m} - G_{(i|\alpha|j)k}P^{\alpha}_{lm} - G_{(i|l\alpha k|}P^{\alpha}_{j)m} - G_{(i|l|j)\alpha}P^{\alpha}_{km}. \tag{27}$$

Let us consider canonical almost geodesic mappings of type π_1 of spaces with affine connections A_n onto generalized 3-Ricci-symmetric spaces $\overline{A}_n$. Hence the Ricci tensor $\overline{R}_{ij}$ for the space $\overline{A}_n$ satisfies the conditions (25). Then the Equation (27) could be written in such form as

$$(n+1)a_{ij,lkm} - a_{lj,ikm} - a_{li,jkm} = -C_{ijlkm}, \tag{28}$$

where

$$C_{ijlkm} = C_{ijlk,m} - G_{\alpha l(j|k|}P^{\alpha}_{i)m} - G_{(i|\alpha|j)k}P^{\alpha}_{lm} - G_{(i|l\alpha k|}P^{\alpha}_{j)m} - G_{(i|l|j)\alpha}P^{\alpha}_{km}. \tag{29}$$

Let us interchange the indices l and j in (28) and symmetrize in the indices i and j. Then we have

$$a_{li,jkm} + a_{lj,ikm} = -\frac{1}{n}C_{(i|l|j)km} + \frac{2}{n}a_{ij,lkm}. \tag{30}$$

Because of (30) the Equation (28) may be written in the form

$$\frac{n^2+n-2}{n}a_{ij,lkm} = -C_{ijlkm} - \frac{1}{n}C_{(i|l|j)km}. \tag{31}$$

The Equation (19) we obtain from the integrability conditions of (4), is to hold for canonical almost geodesic mappings of type π_1 of spaces with affine connections onto generalized 3-Ricci-symmetric spaces.

In the following we have assumed that a space A_n with an affine connection is given. Then taking account of the structure of the tensor C_{ijlkm} which was determined by formulas (4), (18) and (29), we see that the left hand side of Equation (31) depends on the unknown functions $P^h_{ij}(x)$, $a_{ij}(x)$, $a_{ij,k}(x)$, $a_{ij,km}(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijk,m}(x)$.

Let us introduce the tensor a_{ijkm} defined by

$$a_{ijk,m} = a_{ijkm}. \tag{32}$$

Because of (32) we can write the Equation (31) in the form

$$\frac{n^2+n-2}{n}\, a_{ijlk,m} = -C_{ijlkm} - \frac{1}{n}\, C_{(i|l|j)km}. \tag{33}$$

Obviously, in the space A_n the Equations (3), (20)–(22), (32) and (33) form a closed system of PDEs of Cauchy type with respect to the functions $P^h_{ij}(x)$, $a_{ij}(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijkm}(x)$, $a_{ijk}(x)$, $a_{ijkm}(x)$. Also the functions must satisfy the algebraic conditions (24).

Hence we have proved the following theorem.

Theorem 2. *In order that a space A_n with an affine connection admits a canonical almost geodesic mapping of type π_1 onto a generalized 3-Ricci-symmetric space $\overline{A}_n$, it is necessary and sufficient that the mixed system of differential equations of Cauchy type in covariant derivatives (3), (20)–(22), (24), (32) and (33) have a solution with respect to the unknown functions $P^h_{ij}(x)$, $a_{ij}(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijkm}(x)$, $a_{ijk}(x)$ and $a_{ijkm}(x)$.*

Corollary 2. *The family of all generalized 3-Ricci-symmetric spaces, which are images of a given space A_n with an affine connection with respect to canonical almost geodesic mappings of type π_1, depends on no more then*

$$\frac{1}{6}\, n(2n-1)(n^2-1) + \frac{1}{2}\, n(1+n)(n^2+n+1)$$

essential parameters.

6. Canonical Almost Geodesic Mappings of Type π_1 of Spaces with Affine Connections onto Generalized m-Ricci-Symmetric Spaces

A space $\overline{A}_n$ with an affine connection is called *generalized m-Ricci-symmetric* if the Ricci tensor $\overline{R}_{ij}$ for the space satisfies the condition

$$\overline{R}_{ij;\rho_1\rho_2\ldots\rho_m} + \overline{R}_{\rho_1 j;i\rho_2\ldots\rho_m} = 0. \tag{34}$$

It is obvious that generalized 2-Ricci-symmetric spaces and generalized 3-Ricci-symmetric spaces are special cases of generalized m-Ricci-symmetric spaces with $m=2$ and $m=3$ respectively.

If we put $m=1$ in (34) we have a generalized Ricci-symmetric space. Let us differentiate covariantly $(m-3)$ times the Equation (27) with respect to the connection of the space A_n. Then express in the left-hand side the covariant derivatives with respect to the connection of the space A_n in terms of the covariant derivatives with respect to the connection of the space $\overline{A}_n$, using the formula [24]

$$\begin{aligned}(\overline{R}_{ij;\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}})_{,\rho_\tau} &= \overline{R}_{ij;\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}\rho_\tau} + P^\alpha_{i\rho_\tau}\overline{R}_{\alpha j;\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}} + \\ &P^\alpha_{j\rho_\tau}\overline{R}_{i\alpha;\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}} + P^\alpha_{\rho_1\rho_\tau}\overline{R}_{ij;\alpha\ldots\rho_{\tau-2}\rho_{\tau-1}} + \cdots + P^\alpha_{\rho_{\tau-1}\rho_\tau}\overline{R}_{ij;\rho_1\ldots\rho_{\tau-2}\alpha}.\end{aligned}$$

Transforming the left-hand side of (34), we obtain

$$\overline{R}_{ij;\rho_1\rho_2\ldots\rho_m} + \overline{R}_{\rho_1 j;i\rho_2\ldots\rho_m} = (n+1)a_{i\rho_1,j\rho_2\ldots\rho_m} - a_{j\rho_1,i\rho_2\ldots\rho_m} - a_{ij,\rho_1\rho_2\ldots\rho_m} + \Omega_{ij\rho_1\rho_2\ldots\rho_m}, \tag{35}$$

where the tensor $\Omega_{ij\rho_1\rho_2\ldots\rho_m}$ depends on the unknown tensors P^h_{ij}, $\overline{R}^h_{ijk}$, $\overline{R}^h_{ijk,m}$, a_{ij}, $a_{ij,\rho_1}, \ldots, a_{ij,\rho_1\ldots\rho_{m-1}}$.

Let us consider canonical almost geodesic mappings of type π_1 of spaces with affine connections A_n onto generalized m-Ricci-symmetric spaces $\overline{A}_n$. Hence the Ricci tensor $\overline{R}_{ij}$ for the space $\overline{A}_n$ satisfies conditions (34). Then Equation (35) could be written in such form as

$$(n+1)a_{i\rho_1,j\rho_2\ldots\rho_m} - a_{j\rho_1,i\rho_2\ldots\rho_m} - a_{ij,\rho_1\rho_2\ldots\rho_m} = -\Omega_{ij\rho_1\rho_2\ldots\rho_m}. \tag{36}$$

Let us interchange the indices j and ρ_1 in (36) and symmetrize in the indices i and ρ_1. Then we have

$$a_{ij,\rho_1\rho_2\dots\rho_m} + a_{j\rho_1,i\rho_2\dots\rho_m} = -\frac{1}{n}\,\Omega_{(i\rho_1)j\rho_2\dots\rho_m} + \frac{2}{n}\,a_{i\rho_1,\,j\rho_2\dots\rho_m}. \tag{37}$$

Taking account of (37), Equation (36) may be written in the form

$$\frac{n^2+n-2}{n}\,a_{i\rho_1,\,j\rho_2\dots\rho_m} = -\Omega_{ij\rho_1\rho_2\dots\rho_m} - \frac{1}{n}\,\Omega_{(i\rho_1)j\rho_2\dots\rho_m}. \tag{38}$$

Let us introduce the tensors $a_{ijkm\rho_1}, \dots, a_{ijkm\rho_1\dots\rho_{m-4}\rho_{m-3}}$ defined by

$$\begin{aligned} a_{ijkm,\rho_1} &= a_{ijkm\rho_1}, \\ &\dots\dots\dots\dots \\ a_{ijkm\rho_1\dots\rho_{m-4},\rho_{m-3}} &= a_{ijkm\rho_1\dots\rho_{m-4}\rho_{m-3}}. \end{aligned} \tag{39}$$

Because of (39) we can write Equation (38) in the form

$$\frac{n^2+n-2}{n}\,a_{i\rho_1 j\rho_2\dots\rho_{m-1},\rho_m} = -\Omega_{ij\rho_1\rho_2\dots\rho_m} - \frac{1}{n}\,\Omega_{(i\rho_1)j\rho_2\dots\rho_m}. \tag{40}$$

Obviously, the right-hand side of Equation (40) depends on the unknown tensors P^h_{ij}, $\overline{R}^h_{ijk}$, $\overline{R}^h_{ijkm}$, a_{ij}, $a_{ij\rho_1}, \dots, a_{ij\rho_1\dots\rho_{m-1}}$.

Hence, in the space A_n Equations (3), (20)–(22), (32), (39) and (40) form a closed system of PDEs of Cauchy type with respect to the functions $P^h_{ij}(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijkm}(x)$, $a_{ij}(x)$, $a_{ij\rho_1}(x), \dots, a_{ij\rho_1\dots\rho_{m-1}}(x)$. Also the functions must satisfy the algebraic conditions (24).

This proves the following theorem.

Theorem 3. *In order that a space A_n with an affine connection admits a canonical almost geodesic mapping of type π_1 onto a generalized m-Ricci-symmetric space $\overline{A}_n$, it is necessary and sufficient that the mixed system of differential equations of Cauchy type in covariant derivatives* (3), (20)–(22), (24), (32), (39) *and* (40) *have a solution with respect to the unknown functions $P^h_{ij}(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijkm}(x)$, $a_{ij}(x)$, $a_{ij\rho_1}(x), \dots, a_{ij\rho_1\dots\rho_{m-1}}(x)$.*

Corollary 3. *The family of all generalized m-Ricci-symmetric spaces, which are images of a given space A_n with an affine connection with respect to canonical almost geodesic mappings of type π_1, depends on no more then*

$$\frac{1}{6}\,n(2n-1)(n^2-1) + \frac{1}{2(n-1)}\,n(1+n)\,(n^m+n^2-n-1)$$

essential parameters.

7. Conclusions

In this paper, we study canonical almost geodesic mappings of the first type of spaces with an affine connection onto generalized m-Ricci-symmetric spaces. These spaces are the essential generalization of the spaces considered in [24]. Ricci tensor is well known from the Einstein equation that describes gravitational fields. The added condition of generalized symmetric spaces is connected with the study of gravitation waves, which makes the topic of the study relevant.

Author Contributions: All authors contributed equally and significantly in writting this manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the grant IGA PrF 2020014 at Palacky University in Olomouc and by the grant of Faculty of Civil Engineering, Brno University of Technology (research project No. FAST-S-20-6294).

Conflicts of Interest: The authors declare no conflicts of interest.

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