COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
56. DIFFERENTIAL GEOMETRY, EGER (HUNGARY), 1989

Geodesic Mappings of Affine-connected Spaces onto Riemannian Spaces*

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It is known that if a space with affine connection A_n admits a geodesic mapping onto a space with affine connection $\overline{A}_n$ then in general with respect to the mapping system of coordinates $x^1, x^2, \dots, x^n$ the objects of the connection of these spaces $\Gamma^h_{ij}(x)$ and $\overline{\Gamma}^h_{ij}(x)$ obey the following relation [1]:

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij} + \delta^h_{(i}\psi_{j)} \tag{1}$$

where δ^h_i — is the Kronecker symbol, $\psi_i(x)$ — is a vector and (i, j) denotes symmetrization with respect to the indices i and j.

If $\psi_i \not\equiv 0$ then the geodesic mapping is called non-trivial. By the equality (1) it is trivial to find all the spaces with affine connection which admit a non-trivial geodesic mapping (NGM) onto a given Riemannian space.

The present paper is devoted to the investigation of the geodesic mappings of affine-connected spaces A_n without torsion onto the Riemannian spaces $\overline{V}_n$. One may see that not every affine-connected space admits a non-trivial geodesic mapping onto a Riemannian space.

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* This paper is in final form and no version of it will be submitted for publication elsewhere.

1.

Theorem 1. *The space with the affine connection A_n admits a non-trivial geodesic mapping onto a Riemannian space $\overline{V}_n$ with the metric tensor $\overline{g}_{ij}(x)$ if and only if the following set of differential equations with covariant derivatives of Cauchy type has a solution with respect to the symmetric tensor $\overline{g}_{ij}$ (*$\det\|\overline{g}_{ij}\| \neq 0$*), the non-zero vector $\psi_i(x)$ and the invariant $\mu(x)$:*

$$\overline{g}_{ij,k} = 2\psi_k\overline{g}_{ij} + \psi_{(i}\overline{g}_{j)k}; \tag{2a}$$

$$n\psi_{i,j} = n\psi_i\psi_j + \mu\overline{g}_{ij} + \overline{g}_{i\alpha}R^{\alpha}_{\beta\gamma j}\overline{g}^{\beta\gamma} - R_{ij} - \frac{2}{n+1}R^{\alpha}_{\alpha ij}; \tag{2b}$$

$$\begin{aligned}(n-1)\mu_{,i} = \overline{g}^{\alpha\beta}\Big\{ & R^{\gamma}_{\alpha\beta i,\gamma} - R_{\alpha i,\beta} + \\ & + \psi_\alpha\left(4R_{\beta i} - \frac{n-5}{n+1}R^{\gamma}_{\gamma\beta i}\right) + 2(n-1)\psi_\gamma R^{\gamma}_{\alpha\beta i} + \\ & + \frac{2}{n+1}R^{\gamma}_{\gamma\alpha[i,\beta]}\Big\}\end{aligned} \tag{2c}$$

where the comma denotes covariant derivative with respect to the space connection A_n, $\overline{g}^{ij}(x)$ are components of the matrix inverted to $\|\overline{g}_{ij}\|$, R^h_{ijk} and R_{ij} are respectively Riemannian and Ricci tensors of the space A_n.

Proof. Suppose that A_n admits NGM onto $\overline{V}_n$ with metric tensor $\overline{g}_{ij}(x)$. Then the connections A_n and $\overline{V}_n$ obey the relation (1) in general with respect to the mapping coordinate system. Taking into account the covariant constancy of $\overline{g}_{ij}$ in $\overline{V}_n$, the conditions are at the same time sufficient for A_n to admit NGM onto $\overline{V}_n$.

Let us consider integrability conditions of the equations (2a)

$$\overline{g}_{\alpha(h}R^{\alpha}_{i)jk} = 2\overline{g}_{hi}\psi_{[jk]} + \overline{g}_{j(h}\psi_{i)k} - \overline{g}_{k(h}\psi_{i)j} \tag{3}$$

where $\psi_{ij} = \psi_{i,j} - \psi_i\psi_j$, and $[ij]$ denote alternation with respect to i and j.

Convolving (3) and $\overline{g}^{hi}$, we get $\psi_{[jk]} = \frac{1}{n+1}R^{\alpha}_{\alpha jk}$. Excluding $\psi_{[jk]}$ from (3), we obtain

$$\overline{g}_{\alpha(h}R^{\alpha}_{i)jk} - \frac{2}{n+1}\overline{g}_{hi}R^{\alpha}_{\alpha jk} = \overline{g}_{j(h}\psi_{i)k} - \overline{g}_{k(h}\psi_{i)j}. \tag{4}$$

After the convolution (4) with g^{ik} one easily obtains the conditions (2b) with $\mu = \psi_{\alpha\beta}\overline{g}^{\alpha\beta}$.

Taking into account $\overline{g}^{ik}\overline{g}_{kj} = \delta^i_j$, it is not difficult to show that the equations (2a) are equivalent to the relations

$$\overline{g}^{ij}_{,k} = -2\psi_k\overline{g}^{ij} - \delta^{(i}_k\psi^{j)} \tag{5}$$

where $\psi^i \equiv \psi_\alpha\overline{g}^{\alpha i}$.

We covariantly differentiate the conditions (2b) with respect to x^k and then alternate the result with respect to the indices j and k taking into account (2a), (2b), (5) and convolve with $\overline{g}^{ik}$, and finally we get equations (2c).

The theorem has been proved. ■

From theorem 1 we may conclude that the set of all Riemannian spaces, the given affine-connected space A_n admits NGM onto, is dependent on $r \leq r_0 = (n+1)(n+2)/2$ parameters.

Finding of all the solutions of (2) requires a consideration of their integrability conditions and differential extensions, which form a set of algebraic equations with respect to the unknown functions $\overline{g}_{ij}$, ψ_i and μ with coefficients from A_n. But this set is not linear and its solution is certainly difficult.

For equiaffine A_n when the vector ψ_i is necessarily gradient i.e. $\psi_i = \psi_{,i}$ the main equations of NGM onto Riemannian spaces $\overline{V}_n$ may be written in the following from

$$\begin{array}{l} a^{ij}_{,k} = \lambda^{(i}\delta^{j)}_k; \qquad n\lambda^i_{,j} = \mu\delta^i_j + a^{i\alpha}R_{\alpha j} - a^{\alpha\beta}R^i_{\alpha\beta j}; \\ (n-1)\mu_{,i} = 2(n+1)\lambda^\alpha R_{\alpha i} + a^{\alpha\beta}(2R_{\alpha i,\beta} - R_{\alpha\beta,i}) \end{array} \tag{6}$$

where $a^{ij} = a^{ji}$, $|a^{ij}| \neq 0$,
with $a^{ij} = e^{2\psi}\overline{g}^{ij}$; $\lambda^i = -e^{2\psi}\overline{g}^{i\alpha}\psi_\alpha$.

The set of equations (6) is linear the integrability conditions and their differential extensions are a set of linear homogeneous algebraic equations with respect to the unknown functions a^{ij}, λ^i and μ.

2.

The number r of substantial parameters, the general solution of (2) depends on, we shall call the degree of mobility of a_n with respect to the geodesic mappings onto Riemannian spaces $\overline{V}_n$ (by analogy with [1]).

It is easy to prove that maximal degree of mobility $r_0 = (n+1)(n+2)/2$ with respect to the geodesic mappings onto Riemannian spaces is admitted by the projective euclidean spaces and only by them.

The following estimation is obtained for a distribution of the degrees of mobility with respect to the geodesic mappings onto Riemannian spaces.

Theorem 2. *The degree of mobility of the spaces with affine connection A_n, different from the projective-euclidean ones, with respect to the geodesic mappings onto Riemannian spaces does not exceed number $n(n-1)/2$.*

The results under discussion are generalisations of analogous theorems of N. S. Sinyukov [1] for the geodesic mappings of Riemannian spaces.

References

[1] SINYUKOV N. S., Geodesic mappings of Riemannian spaces. Moscow, Nauka, 1979.

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