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SHORT COMMUNICATIONS

# Geodesic Mappings of Equiaffine and Ricci Symmetric Spaces

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## 1. INTRODUCTION

In 1865, E. Beltrami considered geodesic mappings of spaces of constant curvature. In 1869, U. Dini solved a general problem concerning geodesic mappings of surfaces. In 1896, T. Levi-Civita considered geodesic mappings of Riemannian spaces and obtained basic equations for such mappings. Noticeably, his study of geodesic mappings was related to the study of equations for the dynamics of mechanical systems. Later, in 1920, H. Weyl extended these equations to geodesic mappings of affine connection spaces. The theory of geodesic mappings was developed by T. Thomas, J. Thomas, H. Weyl, L. P. Eisenhart, P. A. Shirokov, A. S. Solodovnikov, N. S. Sinyukov, A. V. Aminova, J. Mikeš and other authors; see [1]–[3].

In this note, we give a tensor criterion for the covectors $\psi_i$ in the basic equations of geodesic mappings of affinely connected spaces to be gradient. In the case where the affine connection space under consideration admits a geodesic mapping onto an affine connection space, we find conditions satisfied by these covectors.

Geodesic mappings and transformations of Ricci symmetric spaces were studied by many authors (see, e.g., [4]–[6]). Mikeš proved [7] that Ricci symmetric (pseudo-)Riemannian spaces different from Einstein spaces do not admit nontrivial geodesics mappings. In [8], Einstein spaces admitting nontrivial geodesics mappings were constructed. Detailed information concerning geodesic and other mappings of generalized symmetric, recurrent, and semisymmetric spaces can be found in [1], [3], [9], [10]. Kaigorodov studied the geometric and physical aspects of these spaces in [11].

We present more general results. We obtain basic equations of geodesic mappings from equiaffine spaces onto equiaffine Ricci symmetric spaces in the form of a closed Cauchy-type system of equations in covariant derivatives. We determine the number of essential parameters on which a general solution of this system depends. Our results generalize those obtained for symmetric and generalized symmetric spaces in [12]–[14].

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A general solution of the closed Cauchy-type system of equations (9) and (10) in covariant derivatives depends on at most $n(n+3)/2$ essential parameters.

Let us find integrability conditions for Eq. (9). To this end, we covariantly differentiate (9) with respect to $x^k$ in the space $\mathbb{A}_n$ and alternate over the indices $j$ and $k$. Using the Ricci identities, we obtain after transformations

$$(n-1)\psi_\alpha R^\alpha_{ijk} = R_{ik,j} - R_{ij,k} + R_{ij}\psi_k - R_{ik}\psi_j. \tag{11}$$

Let us find integrability conditions for Eq. (10). To this end, we covariantly differentiate (10) with respect to $x^l$ in the space $\mathbb{A}_n$ and perform alternating over the indices $k$ and $l$. Using the Ricci identities and performing transformations, we obtain

$$\overline{R}_{\alpha j} R^\alpha_{ikl} + \overline{R}_{i\alpha} R^\alpha_{jkl} = \frac{1}{n-1}(R_{ik}\overline{R}_{lj} + R_{jk}\overline{R}_{il} - R_{il}\overline{R}_{kj} - R_{jl}\overline{R}_{ik}). \tag{12}$$

Obviously, the equations (11) and (12) are linear in the unknowns tensors $\psi_i$ and $\overline{R}_{ij}$. In the case where the space $\overline{\mathbb{A}}_n$ is flat, an integrability condition for Eqs. (9) and (10) is Eq. (11), because (12) turns into an identity.

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