Slovak University of Technology in Bratislava
Faculty of Mechanical Engineering
19th Conference on Applied Mathematics **APLIMAT 2020**

Proceedings

CONFORMAL AND GEODESIC MAPPINGS ONTO RICCI SYMMETRIC SPACES

BEREZOVSKII Volodymyr (UA), MIKEŠ Josef (CZ), RÝPAROVÁ Lenka (CZ)

Abstract. In this paper, we consider the conformal and geodesic mappings onto Ricci symmetric spaces. We obtained fundamental equations in the Cauchy type form, which depend on finite real parameters.

Keywords: conformal mapping, geodesic mapping, Ricci symmetric space, fundamental equations, Cauchy type differential equations

Mathematics subject classification: Primary 53B20; Secondary 53B22

1 Introduction

Conformal mappings of Riemannian spaces have been reviewed in many papers. These mappings have significant applications in the general theory of relativity (for example, [4, 5, 13, 14, 15]).

Further, we assume that the metric of Riemannian spaces is arbitrary, i.e. these spaces are Riemannian or (pseudo-) Riemannian.

The question is whether the Riemannian space admits or does not allow conformal mapping onto some Einstein space, reduced by H. Brinkmann [3] to the problem of the existence of a solution to some nonlinear system of Cauchy-type differential equations with respect to unknown functions. This task is described in detail in the monograph by A.Z. Petrov [13].

In the papers [1, 6, 7, 12], the main equations of these mappings were reduced to a linear system of differential equations in covariant derivatives of Cauchy type, with the help of which it was possible to estimate the degree of parametric arbitrariness in the general solution of this problem. That is, it was possible to establish the degree of mobility of Riemannian spaces with respect to conformal mappings onto Einstein spaces. In [12], an estimate was obtained of the first lacuna in the distribution of degrees of mobility of Riemannian spaces, with respect to conformal mappings, onto Einstein spaces.

As is known [12], conformally flat Riemannian spaces admit maximum values of degrees of mobility and only they. A criterion in tensor is obtained for spaces other than conformally flat Riemannian spaces for which the maximum possible degree of mobility is $r = n - 1$, where n is the dimension of the spaces in question $(n > 2)$. Hence, the estimation of the first lacuna in a distribution of degrees of mobility of Riemannian spaces with respect to conformal mappings onto Einstein spaces is obtained, and maximally mobile spaces are distinguished from conformally flat Riemannian spaces with respect to the indicated degrees of mobility.

The paper [6] presents the minimal conditions on the differentiability of geometric objects under consideration to be satisfied by conformal mappings of Riemannian spaces V_n onto Einstein spaces. The main equations for the mappings are obtained as a closed linear system in covariant derivatives of Cauchy-type taking into account the minimal requirements on the differentiability of metrics of spaces which are conformally equivalent.

The theory of geodesic mappings ideologically goes back to the work of Levi-Civita [9]. He posed and solved in a special coordinate system the problem of search the proper Riemannian spaces with common geodesics. It is noteworthy that it was related with the study of the equations of dynamics of mechanical systems. Then the theory of geodesic mappings was developed in the works of Thomas, Weyl, Shirokov, Solodovnikov, Sinyukov, Mikeš and others, see [5, 11, 10, 13, 16].

The most famous equations are the Levi-Civita equations obtained by Levi-Civita himself for the case of Riemannian spaces. Later, H. Weyl obtained the same equations for geodesic mappings between spaces with affine connections. N.S. Sinyukov [16] (see [10, 11]) proved that the main equations of geodesic mappings of (pseudo)-Riemannian spaces are equivalent to some linear system of equations of Cauchy-type in covariant derivatives.

In [2], these results are generalized to the case of geodesic mappings of equiaffine spaces with affine connections onto (pseudo)-Riemannian spaces.

In this paper, the main equations of conformal and geodesic mappings of Riemannian spaces onto Ricci symmetric Riemannian spaces are obtained in the form of closed-systems of Cauchy-type equations in covariant derivatives.

We established the number of essential parameters on which the general solutions of the found systems of equations of Cauchy-type in covariant derivatives depend.

We suppose that all geometric objects under consideration are continuous and sufficiently smooth.

2 Basic concepts of theories of conformal and geodesic mappings

Consider the map f of the Riemannian space V_n with the metric tensor g onto the Riemannian space $\bar{V}_n$ with the metric tensor $\bar{g}$.

Assume that the Riemannian spaces V_n and $\bar{V}_n$ assigned to the common coordinate system $x = (x^1, x^2, \ldots, x^n)$.

Mapping f: $V_n \to \bar{V}_n$ is called *conformal* if, in the general mapping f and in the coordinate system x, metric tensors g and $\bar{g}$ are proportional and for the components of metric tensors there is a dependence

$$\bar{g}_{ij}(x) = e^{2\psi(x)} \cdot g_{ij}(x), \tag{1}$$

where ψ is a function.

From (1) it follows that under conformal mapping the angles between the tangent vectors of the curves are preserved. Conformal mappings are fully characterized by this property.

From (1) the following relationship between the Christoffel symbols of the second kind of spaces V_n and $\bar{V}_n$ follows

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_j \psi_i(x) + \delta^h_i \psi_j(x) - \psi^h(x) g_{ij}(x), \tag{2}$$

where $\psi_i = \frac{\partial \psi}{\partial x^i}$ is a gradient vector, $\psi^h = g^{h\alpha}\psi_\alpha$, $g^{ij}$are the components of the inverse matrix to the matrix g_{ij}, δ^h_i is the Kronecker symbols.

A conformal mapping is called *homothetic* if the function $\psi(x)$ is constant, i.e. $\bar{g}_{ij}(x) = c \cdot g_{ij}(x)$. This condition is equivalent to $\psi_i(x) = 0$, therefore, such a mapping is also affine.

Recall that in a Riemannian space V_n with the metric tensor $g_{ij}(x)$ we define the Riemann tensor, Ricci tensor, and scalar curvature as follows

$$R^h_{ijk} = \frac{\partial \Gamma^h_{ik}}{\partial x^j} - \frac{\partial \Gamma^h_{ij}}{\partial x^k} + \Gamma^h_{ik}\Gamma^h_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^h_{\alpha k}, \qquad R_{ij} = R^\alpha_{ij\alpha}, \qquad R = R_{\alpha\beta} g^{\alpha\beta}. \tag{3}$$

It is known [5, 10, 13, 16] that under conformal mappings the Riemann tensors of the spaces V_n and $\bar{V}_n$ are related by

$$\bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik} + g_{ij}\psi^h_k - g_{ik}\psi^h_j + \left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right) \cdot \Delta_1 \psi, \tag{4}$$

where $\psi_{ij} = \psi_{i,j} - \psi_i \psi_j$, $\psi^h_k = g^{h\alpha}\psi_{\alpha k}$, $\Delta_1 \psi = g^{\alpha\beta}\psi_\alpha \psi_\beta$, the sign ", " means covariant differentiation in V_n.

Contracting (4) by the indices h and k, after transformations we get

$$\psi_{i,j} = \frac{\mu}{n-2} g_{ij} + \psi_i \psi_j - \frac{1}{n-2}\left(\bar{R}_{ij} - R_{ij}\right), \tag{5}$$

where μ is a function.

A curve defined in the space of affine connection A_n is called a *geodesic* if its tangent vector is parallel along it.

A mapping $f\colon A_n \to \bar{A}_n$ is called a *geodesic* if any geodesic of the space A_n maps onto the geodesic of the space $\bar{A}_n$.

It is known [10, 11, 13, 16] that in order for the map f of the space A_n to the space $\bar{A}_n$ to be geodesic, it is necessary and sufficient that in the coordinate system $(x^1, x^2, \ldots, x^n)$ deformation tensor of connection

$$P^h_{ij}(x) = \bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x), \tag{6}$$

is presented as

$$P^h_{ij}(x) = \psi_i(x)\delta^h_j + \psi_j(x)\delta^h_i, \tag{7}$$

where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are components of connections A_n and $\bar{A}_n$, $\psi_i(x)$ is a vector.

A geodesic map is called nontrivial if $\psi_i(x) \neq 0$. Obviously, any space A_n with affine connection admits a nontrivial geodesic mapping onto some other space $\bar{A}_n$ with affine connection. Generally speaking, such an assumption is not true with respect to geodesic mappings of Riemannian spaces onto Riemannian spaces. In particular, Riemannian spaces were distinguished that prevent geodesic mappings on Riemannian spaces.

3 Conformal mappings of Riemannian spaces to Ricci symmetric spaces

An affinely connected or Riemannian space is called *Ricci symmetric* if the Riemann tensor in it is absolutely parallel. Thus, the Ricci symmetric spaces $\bar{A}_n$ $(\bar{V}_n)$ are characterized by the condition

$$\bar{R}_{ij|k} = 0, \tag{8}$$

where sign "|" denotes the covariant derivative in $\bar{A}_n(\bar{V}_n)$, $\bar{R}_{ij}$ are components of Ricci tensor of space $\bar{A}_n(\bar{V}_n)$.

If the Riemannian spaces V_n and $\bar{V}_n$ are assigned to the coordinate map $x = (x^1, \dots, x^n)$, then the condition (8) by definition, the covariant derivative can be written in the following form

$$\bar{R}_{ij,k} = 2\psi_k \bar{R}_{ij} + \psi_i \bar{R}_{jk} + \psi_j \bar{R}_{ik} - \psi^\alpha \bar{R}_{i\alpha} g_{jk} - \psi^\alpha \bar{R}_{j\alpha} g_{ik}. \tag{9}$$

It is easy to verify that if the invariant $\psi(x)$ in V_n, the generating gradient vector $\psi_i(x)$, and the symmetric tensor $\bar{R}_{ij}(x)$ are a solution of equations (5) and (9), then under the conformal mapping (1) of the space V_n onto the space $\bar{V}_n$, by if necessary, the space $\bar{V}_n$ is Ricci symmetric and the tensor $\bar{R}_{ij}(x)$ is the Ricci tensor of this space.

Conditions (5) and (9) occur only when

$$g_{ij}(x) \in C^2, \quad \psi(x) \in C^2, \quad \psi_i(x) \in C^1, \quad \mu \in C^0, \quad R_{ij}(x) \in C^0, \quad \bar{R}_{ij}(x) \in C^1. \tag{10}$$

It follows that $\bar{g}_{ij}(X) \in C^2$.

It is easy to verify that in the case when $R_{ij} \in C^1$, then $\psi \in C^3$, $\psi_i \in C^2$ and $\mu \in C^1$. This follows from a modification of the formula (9) and the universal Lemma formulated and proved in [8], where it is shown that if the equality $\partial_i \lambda^h(x) - \mu(x)\delta_i^h \in C^1$, then $\lambda^h(x) \in C^2$ and $\mu(x) \in C^1$.

Naturally, the equation holds

$$\psi_{,i} = \psi_i. \tag{11}$$

We differentiate (5) with respect to x^k in the Riemannian space V_n, and then we alter through the indices j and k. Given the Ricci identity and the fact that the Ricci tensor is symmetric, after the transformations we get

$$\begin{aligned}(n-2)\psi_\alpha R^\alpha_{ijk} \;=\;& -g_{ij}\mu_{,k} + g_{ik}\mu_{,j} - g^{\alpha\beta}\psi_\alpha \left(g_{ik}\bar{R}_{\beta j} - g_{ij}\bar{R}_{\beta k}\right) + \\ & R_{ik,j} - R_{ij,k} + R_{ij}\psi_k - R_{ik}\psi_j + \mu(g_{ij}\psi_k - g_{ik}\psi_j).\end{aligned} \tag{12}$$

Then we contract (12) with g^{ij} and use the Foss-Weil formula $R_{ij,k}g^{jk} = (1/2)R_{,i}$. As a result, we obtain the equation

$$(n-1)\mu_{,k} = g^{\alpha\beta}\Big[(n-2)\psi_\gamma R^\gamma_{\beta k\alpha} - (n-1)\psi_\beta \bar{R}_{\alpha k} - \psi_\beta R_{\alpha k}\Big] + \Big[R + (n-1)\mu\Big]\psi_k - \frac{1}{2}R_{,k}. \tag{13}$$

Obviously, the equations (5), (9), (11) and (13) in this space V_n are closed Cauchy-type system with respect to the functions $\psi(x)$, $\psi_i(x)$, $\mu(x)$ and $\bar{R}_{ij}(x)$, and, of course, the conditions of algebraic character $\bar{R}_{ij}(x) = \bar{R}_{ji}(x)$.

This proves

Theorem 1 *In order for the Riemannian space V_n to conform to the Ricci conformal symmetric Riemannian space $\bar{V}_n$, it is necessary and sufficient that it contains a solution of a closed system of equations in covariant derivatives of Cauchy type (5), (9), (11) and (13) with respect to unknown functions $\psi(x)$, $\psi_i(x)$, $\mu(x)$ and $\bar{R}_{ij}(x)(= \bar{R}_{ji}(x))$.*

Thus, the general solution of the above system of differential equations depends on

$$(1/2)n(n+1) + n + 2$$

the initial values of unknown functions at some point x_0:

$$\psi(x_0), \psi_i(x_0), \mu(x_0) \quad \text{and} \quad \bar{R}_{ij}(x_0)\ (= \bar{R}_{ji}(x_0)),$$

which, in the general case, are interdependent.

4 Geodesic mappings of spaces of affine connection on Ricci symmetric spaces

Consider the geodesic mappings of affine connected spaces A_n on Ricci symmetric spaces $\bar{A}_n$. Suppose that the spaces A_n and $\bar{A}_n$ are assigned to a coordinate system common in the map.

Since, by definition,

$$\bar{R}^h_{ijk|m} = \frac{\partial \bar{R}^h_{ijk}}{\partial x^m} + \bar{\Gamma}^h_{m\alpha}\bar{R}^\alpha_{ijk} - \bar{\Gamma}^\alpha_{mi}\bar{R}^h_{\alpha jk} - \bar{\Gamma}^\alpha_{mj}\bar{R}^h_{i\alpha k} - \bar{\Gamma}^\alpha_{mk}\bar{R}^h_{ij\alpha},$$

then given the formula (6), we can write

$$\bar{R}^h_{ijk|m} = \bar{R}^h_{ijk,m} + P^h_{m\alpha}\bar{R}^\alpha_{ijk} - P^\alpha_{mi}\bar{R}^h_{\alpha jk} - P^\alpha_{mj}\bar{R}^h_{i\alpha k} - P^\alpha_{mk}\bar{R}^h_{ij\alpha}. \tag{14}$$

Let us contract (14) by the indices h and k. As a result, we obtain

$$\bar{R}_{ij|m} = \bar{R}_{ij,m} - P^\alpha_{mi}\bar{R}_{\alpha j} - P^\alpha_{mj}\bar{R}_{i\alpha}. \tag{15}$$

Since the space $\bar{A}_n$ Ricci is symmetric, the formula (8) holds, therefore

$$\bar{R}_{ij,m} = P^\alpha_{mi}\bar{R}_{\alpha j} + P^\alpha_{mj}\bar{R}_{i\alpha}. \tag{16}$$

In fact, the formula (16) holds for mappings of any nature with affine connected spaces to Ricci symmetric spaces.

Considering that the connection strain tensor $P^h_{ij}(x)$ has the structure (7), on the basis of the formula (16) we have

$$\bar{R}_{ij,m} = 2\psi_m\bar{R}_{ij} + \psi_i\bar{R}_{mj} + \psi_j\bar{R}_{im}. \tag{17}$$

It is known [10, 16] that between the Riemann tensors R^h_{ijk}, $\bar{R}^h_{ijk}$ affine spaces A_n and $\bar{A}_n$ respectively, there is a dependency

$$\bar{R}^h_{ijk} = R^h_{ijk} + P^h_{ik,j} - P^h_{ij,k} + P^\alpha_{ik}P^h_{\alpha j} - P^\alpha_{ij}P^h_{\alpha k}. \tag{18}$$

Given that

$$P^h_{ij,k} = \psi_{i,k}\delta^h_j + \psi_{j,k}\delta^h_i,$$

from the formula (18) after the transformations we get

$$\bar{R}^h_{ijk} = R^h_{ijk} - \delta^h_j \psi_{i,k} + \delta^h_k \psi_{i,j} - \delta^h_i \psi_{j,k} + \delta^h_i \psi_{k,j} + \delta^h_j \psi_i \psi_k - \delta^h_k \psi_i \psi_j. \tag{19}$$

Let us contract (19) by the indices h and k. As a result, we find

$$\bar{R}_{ij} = R_{ij} + h\psi_{i,j} - \psi_{j,i} + (1-n)\psi_i \psi_j. \tag{20}$$

The equation (20) is alternatable with respect to the indices i and j. We have

$$\bar{R}_{[ij]} = R_{[ij]} + (n+1)\psi_{i,j} - (n+1)\psi_{j,i}, \tag{21}$$

where $[ij]$ denotes alternation at indices i and j.

From the equation (21) we find

$$\psi_{i,j} - \psi_{j,i} = \frac{1}{n+1}\left(\bar{R}_{[ij]} - R_{[ij]}\right). \tag{22}$$

From the equation (20), taking into account the equation (22), we have

$$\psi_{i,j} = \frac{1}{n^2-1}\left[n\bar{R}_{ij} + \bar{R}_{ji} - (nR_{ij} + R_{ji})\right] + \psi_i \psi_j. \tag{23}$$

Obviously, the equations (17) and (23) in this space A_n are a closed Cauchy system with respect to the functions $\bar{R}_{ij}(x)$ and $\psi_i(x)$.

Thereby proved

Theorem 2 *In order for the space of affine connection A_n to allow a geodesic mapping onto the Ricci symmetric space $\bar{A}_n$, it is necessary and sufficient that it contains a solution of a closed system of equations in covariant derivatives of the Cauchy type (17), (23) with respect to the functions $\bar{R}_{ij}(x)$ and $\psi_i(x)$.*

The general solution of a closed system of equations in covariant derivatives of Cauchy type (17), (23) depends on no more than $n\,(n+1)$ of essential parameters.

The integrability conditions for the equations (17) and (23), respectively, are of the form

$$\begin{gathered}(n^2-1)\big(\bar{R}_{\alpha j}R^\alpha_{ikm} + \bar{R}_{i\alpha}R^\alpha_{ikm}\big) - 2(n-1)\big(\bar{R}_{km} - \bar{R}_{mk}\big)\bar{R}_{ij} - \\ -n\big(\bar{R}_{kj} - \bar{R}_{jk}\big)\bar{R}_{im} - \big(\bar{R}_{mi} - \bar{R}_{im}\big)\bar{R}_{kj} - n\big(\bar{R}_{jm} - \bar{R}_{mj}\big)\bar{R}_{ki} - \\ -\big(\bar{R}_{ik} - \bar{R}_{ki}\big)\bar{R}_{mj} + \big(nR_{jm} + R_{mj}\big)\bar{R}_{ik} + \big(nR_{im} + R_{mi}\big)\bar{R}_{kj} - \\ -\big(nR_{ik} + R_{ki}\big)\bar{R}_{mj} - \big(nR_{jk} + R_{kj}\big)\bar{R}_{im} = 0,\end{gathered} \tag{24}$$

$$\begin{aligned}(n^2-1)\psi_\alpha R^\alpha_{ijk} + n\psi_j R_{ik} + \psi_j R_{ki} - n\psi_k R_{ij} - \psi_k R_{ji} + (n-1)\psi_i\big(R_{jk} - R_{kj}\big) = \\ = n\big(R_{ik,j} - R_{ij,k}\big) + R_{ki,j} - R_{ji,k}.\end{aligned} \tag{25}$$

Obviously, the equation (25) is independent of $\bar{R}_{ij}$ and linear with respect to ψ_i. The equation (24) is independent of ψ_i and when the space $\bar{A}_n$ is equiaffine, it becomes linear with respect to $\bar{R}_{ij}$.

The space $\bar{A}_n$ is called equiaffine if the condition Ricci tensor of this space is satisfied

$$\bar{R}_{ij} = \bar{R}_{ji}.$$

Thus, the integrability conditions for the equations (17) and (23) of geodesic mappings of spaces of affine connection A_n to Ricci, the symmetric equiaffine spaces $\bar{A}_n$ will be linear with respect to unknowns functions $\psi_i(x)$ and $\bar{R}_{ij}(x)$.

The paper was supported by the project IGA PrF 2019015 Palacky University Olomouc.

References

[1] BEREZOVSKII, V., HINTERLEITNER, I., GUSEVA, N., MIKEŠ, J., Conformal mappings of Riemannian spaces onto Ricci symmetric spaces, *Math. Notes*, 103(2), 2018:304–307.

[2] BEREZOVSKII, V., HINTERLEITNER, I., MIKEŠ, J., Geodesic mappings of manifolds with affine connection onto the Ricci symmetric manifolds, *Filomat*, 32(2), 2018:379–385.

[3] BRINKMANN, H., Einstein spaces which mapped conformally on each other, *Math. Ann.*, 94, 1925.

[4] DENISOV, V. I., Special conformal mappings in general relativity, *Ukrain. Geom. Sb.*, (28), 1985:43–50, ii.

[5] EISENHART, L., Non-Riemannian geometry. Princeton Univ. Press. 1926, *AMS Colloq. Publ.*, 8, 2000.

[6] EVTUSHIK, L. E., HINTERLEITNER, I., GUSEVA, N. I., MIKEŠ, J., Conformal mappings onto Einstein spaces, *Russ. Math.*, 60(10), 2016:5–9.

[7] HINTERLEITNER, I., GUSEVA, N., MIKEŠ, J., On conformal mappings onto compact Einstein manifolds, *Geometry, Integrability and Quantization*, 19, 2018:132–139.

[8] HINTERLEITNER, I., MIKEŠ, J., Geodesic mappings and Einstein spaces, *Trends Math.*, 19, 2013:331–335.

[9] LEVI-CIVITA, T., Sulle transformationi dello equazioni dinamiche, *Ann. Mat. Milano*, 24(2), 1896:255–300.

[10] MIKEŠ, J., ET AL., *Differential geometry of special mappings*, Palacky Univ. Press, Olomouc, 2015.

[11] MIKEŠ, J., VANŽUROVÁ, A., HINTERLEITNER, I., *Geodesic mappings and some generalizations*, Palacky Univ. Press, Olomouc, 2009.

[12] MIKEŠ, U., GAVRILCHENKO, M. L., GLADYSHEVA, E. I., Conformal mappings onto Einstein spaces, *Mosc. Univ. Math. Bull.*, 49(3), 1994:10–14.

[13] PETROV, A. Z., *New Methods in the Theory of Relativity*, Nauka, Moscow, 1965.

[14] SCHOUTEN, J., STRUIK, D., *Einfuehrung in die Neueren Methoden der Differentialgeometrie. B. 1*, Noordhoff, Groningen, 1935; Gostekhizdat, Moscow-Leningrad, 1939.

[15] SCHOUTEN, J., STRUIK, D., *Einfuehrung in die Neueren Methoden der Differentialgeometrie. B. 2*, Noordhoff, Groningen, 1938; Gostekhizdat, Moscow-Leningrad, 1949.

[16] SINYUKOV, N., *Geodesic mappings of Riemannian spaces*, Nauka, Moscow, 1979.

Current address

Berezovskii Volodymyr, prof., CSc.
Department of Mathematics
Uman National University of Horticulture, Uman, Ukraine
E-mail: berez.volod@rambler.ru

Mikeš Josef, prof. RNDr., DrSc.
Department of Algebra and Geometry
Faculty of Science
Palacky University in Olomouc
17. listopadu 12, 771 46 Olomouc, Czech Republic
E-mail: josef.mikes@upol.cz

Rýparová Lenka, Mgr.
Department of Algebra and Geometry
Faculty of Science
Palacky University in Olomouc
17. listopadu 12, 771 46 Olomouc, Czech Republic
E-mail: lenka.ryparova01@upol.cz