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ON A CLASS OF CURVATURE PRESERVING  
ALMOST GEODESIC MAPPINGS OF MANIFOLDS  
WITH AFFINE CONNECTION

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**Abstract.** In this paper we pay attention to a particular case of almost geodesic mappings of the first type between (differentiable) manifolds with affine connection. We use here classical tensor methods and the apparatus of partial differential equations.

We prove that under the mappings under consideration, the invariant geometric object is just the (Riemannian) curvature tensor of the connection. We present the basic equations of the class of mappings under consideration in an equivalent form of the Cauchy system in covariant derivatives.

**Key words and phrases.** almost geodesic mappings, invariant geometric object, manifolds with affine connection.

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## 1 Introduction

Many monographs and papers are devoted to the theory of geodesic and holomorphically projective mappings, see [1]-[23]. We continue here a research project on geodesic and almost geodesic mappings of spaces with affine connection, or pseudo-Rimannian spaces, respectively.

In this paper we pay attention to a particular case of almost geodesic mappings of the first type between (differentiable) manifolds endowed affine connection. We prove that under such maps, the invariant geometric object is just the (Riemannian) curvature tensor of the connection. We present the basic equations of the class of mappings under consideration in an equivalent form of the Cauchy system in covariant derivatives.

As the main tool, we use here classical tensor methods and the apparatus of partial differential equations.

## 2 Almost geodesic mappings of manifolds with affine connection

Let us recall the basic concepts of the theory of almost geodesic mappings of manifolds with affine connection introduced in [16, 17, 18]. A (differentiable) curve defined in a manifold with affine connection  $A_n$  is called *almost geodesic* if there is a (differentiable) two-dimensional parallel distribution along the curve such that the tangent vectors of the curve, being parallelly transported along the curve, still belong to the distribution.

A diffeomorphism  $f: A_n \rightarrow \bar{A}_n$  of manifolds with affine connection is called *almost geodesic* if all geodesics in  $A_n$  are mapped onto almost geodesic curves of  $\bar{A}_n$ .

A map of  $A_n$  onto  $\bar{A}_n$  is almost geodesic if and only if in a common coordinate system  $(x^1, \dots, x^n)$  (with respect to the diffeomorphism  $f$ , [15, p. 85]), the deformation tensor of the connections ([15, p. 86])  $P_{ij}^h(x) = \bar{\Gamma}_{ij}^h(x) - \Gamma_{ij}^h(x)$  satisfies

$$A_{\alpha\beta\gamma}^h \lambda^\alpha \lambda^\beta \lambda^\gamma \equiv a P_{\alpha\beta}^h \lambda^\alpha \lambda^\beta + b \lambda^h$$

where  $A_{ijk}^h \equiv P_{ij,k}^h + P_{ij}^\alpha P_{\alpha k}^h$ ,  $\Gamma_{ij}^h$  ( $\bar{\Gamma}_{ij}^h$ , respectively) are components of the connection in the manifold  $A_n$  ( $\bar{A}_n$ , respectively),  $\lambda^h$  is an arbitrary vector, and  $a, b$  are some functions of the variables  $x^h, \lambda^h$ . Here and in what follows, “,” denotes the covariant derivative with respect to the connection of  $A_n$ .

N.S. Sinyukov distinguished in [12, 16, 17, 18] three types of almost geodesic mappings denoted by  $\pi_1, \pi_2$  and  $\pi_3$ . We proved in [1, 12] that for dimensions  $n > 5$ , there are no others. Almost geodesic mappings (in short, AGM) of type  $\pi_1$  are characterized by the following conditions for the deformation tensor

$$A_{(ijk)}^h = \delta_{(i}^h a_{jk)} + b_{(i} P_{jk)}^h$$

where  $a_{ij}$  is a symmetric tensor,  $b_i$  is a covector,  $\delta_i^h$  is the Kronecker tensor, and  $(ijk)$  means symmetrization (without division) with respect to the listed indices.

## 3 A particular subclass of the first type AGM

Let then following conditions are satisfied under the diffeomorphism of manifolds with affine connection:

$$P_{ij,k}^h = -P_{ij}^\alpha P_{\alpha k}^h + \delta_{(k}^h a_{ij)} \tag{1}$$

Such mappings belong, as a particular case, to AGM of the first type. The Riemannian tensors  $R_{ijk}^h$  and  $\bar{R}_{ijk}^h$  of the manifolds  $A_n$  and  $\bar{A}_n$ , respectively, are related by [17]

$$\bar{R}_{ijk}^h = R_{ijk}^h + P_{i[k,j]}^h + P_{i[k}^\alpha P_{j]\alpha}^h \tag{2}$$

where  $[kj]$  denotes alternation with respect to the distinguished indices. According to (1) and (2), the following holds:

**Theorem 3.1** *The Riemannian tensor  $R_{ijk}^h$  is an invariant object of manifolds with affine connection under almost geodesic mappings satisfying (1).*

Since the Riemmanian tensor vanishes in affine spaces, we deduce that the affine spaces form a class closed under AGM satisfying (1):

**Theorem 3.2** *If an affine space  $A_n$  admits an almost geodesic mapping onto  $\bar{A}_n$  that satisfies (1), then  $\bar{A}_n$  is an affine space.*

Regarding (1) as a system of Cauchy type with respect to the components of the deformation tensor  $P_{ij}^h$  we find the corresponding integrability conditions. For this purpose, let us calculate covariant derivatives of (1) with respect to  $x^m$ , and let us alternate in  $k$  and  $m$ . Accounting the Ricci identity we get

$$\delta_i^h a_{j[k,m]} + \delta_j^h a_{i[k,m]} + \delta_{[k}^h a_{i,j|m]} = -P_{ij}^\alpha R_{\alpha km}^h + P_{\alpha(j}^h R_{i)km}^\alpha + a_{j|m} P_{k|i}^h + a_{i|m} P_{k|j}^h. \quad (3)$$

Now contracting the integrability conditions in  $h$  and  $m$  we get

$$a_{jk,i} + a_{ik,j} - (n+1)a_{ij,k} = -P_{ij}^\alpha R_{\alpha k} + P_{\alpha j}^\beta R_{ik\beta}^\alpha + P_{\alpha i}^\beta R_{jk\beta}^\alpha + a_{j\alpha} P_{ki}^\alpha - a_{jk} P_{\alpha i}^\alpha + a_{i\alpha} P_{jk}^\alpha - a_{ik} P_{j\alpha}^\alpha. \quad (4)$$

Further, alternating (4) over  $k$  and  $j$  we obtain

$$a_{ij,k} = a_{ik,j} + \frac{1}{n+2} (P_{ij}^\alpha R_{\alpha k} - P_{ik}^\alpha R_{\alpha j} - P_{\alpha j}^\beta R_{ik\beta}^\alpha - P_{\alpha i}^\beta R_{jk\beta}^\alpha + P_{\alpha k}^\beta R_{ij\beta}^\alpha + P_{\alpha i}^\beta R_{kj\beta}^\alpha - a_{j\alpha} P_{ki}^\alpha + a_{k\alpha} P_{ij}^\alpha + a_{ik} P_{j\alpha}^\alpha - a_{ij} P_{k\alpha}^\alpha). \quad (5)$$

In (5), let us interchange the indices  $k$  and  $i$ ,

$$a_{kj,i} = a_{ik,j} + \frac{1}{n+2} (P_{kj}^\alpha R_{\alpha i} - P_{ki}^\alpha R_{\alpha j} - P_{\alpha j}^\beta R_{ki\beta}^\alpha - P_{\alpha k}^\beta R_{ji\beta}^\alpha + P_{\alpha i}^\beta R_{kj\beta}^\alpha + P_{\alpha k}^\beta R_{ij\beta}^\alpha - a_{j\alpha} P_{ik}^\alpha + a_{i\alpha} P_{kj}^\alpha + a_{ki} P_{j\alpha}^\alpha - a_{kj} P_{i\alpha}^\alpha). \quad (6)$$

Plugging (5) and (6) to (4) we find

$$a_{ik,j} = \frac{1}{(n-1)(n+2)} [n(P_{ik}^\alpha R_{\alpha j} - P_{\alpha(k}^\beta R_{i)j\beta}^\alpha) + R_{\alpha(k} P_{i)j}^\alpha - P_{\alpha j}^\beta R_{(ik)\beta}^\alpha - P_{\alpha(i}^\beta R_{|j|k)\beta}^\alpha + (n+1)(a_{j(i} P_{k)\alpha}^\alpha - a_{\alpha(i} P_{k)j}^\alpha) + 2(a_{ik} P_{j\alpha}^\alpha - a_{j\alpha} P_{ik}^\alpha)]. \quad (7)$$

Obviously, the equations (1) and (7) in the given space  $A_n$  represent a Cauchy system in the functions  $P_{ij}^h(x)$  and  $a_{ij}(x)$  which, naturally, must satisfy also the following system of conditions of an algebraic character

$$P_{ij}^h(x) = P_{ji}^h(x), \quad a_{ij}(x) = a_{ji}(x). \quad (8)$$

Hence we have proved:

**Theorem 3.3** *A manifold with affine connection  $A_n$  admits almost geodesic mappings, satisfying the equation (1), onto a manifold with affine connection  $\bar{A}_n$  if and only if there exists, in  $A_n$ , a solution of the mixed system of Cauchy type (1), (7) and (8) in the functions  $P_{ij}^h$  and  $a_{ij}$ .*

It is proved that the number of relevant parameters on which the solution of a system under consideration depends has the upper boundary

$$r \leq \frac{1}{2} n(n+1)^2.$$

#### 4 An example of a particular subclass of the first type AGM

It the tensor  $a_{ij}$  vanishes identically the equation (1) reads

$$P_{ij,k}^h = -P_{ij}^\alpha P_{\alpha k}^h. \quad (9)$$

The equations (9) are completely integrable in a manifolds with affine connection. That is, the system is solvable for any initial conditions  $P_{ij}^h(x_0)$ . If we choose initial values satisfying  $P_{ij}^h(X_0) \neq \delta_{(i}^h \psi_{j)}(x_0)$  then the obtained solution determines an almost geodesic map of the first type of an affine space  $A_n$  onto an affine space  $\bar{A}_n$  that is not a geodesic one. Hence we have as a consequence

**Theorem 4.1** *There exists an almost geodesic map of the first type of the affine space onto itself under which all straight lines are mapped onto plane curves not all of which are straight lines.*

Let  $(x^1, \dots, x^n)$  and  $(\bar{x}^1, \dots, \bar{x}^n)$  be affine coordinates in affine spaces  $A_n$  and  $\bar{A}_n$ , respectively. We give here a particular example of an almost geodesic map of the first type of a flat space  $A_n$  onto a flat space  $\bar{A}_n$  as follows. Pointwise, the map is given in coordinates by

$$\bar{x}^h = \frac{1}{2} C_\alpha^h (x^\alpha - C^\alpha)^2 + x_0^h, \quad (10)$$

where  $C_i^h, C^h, x_0^h$  are constants such that  $x^h \neq C^h, \det(C_i^h) \neq 0$ . It can be checked directly that the only non-zero components of the deformation tensor are

$$P_{ii}^i = \frac{1}{x^i - C^i}, \quad i = 1, \dots, n.$$

It can be verified that the tensor  $P_{ij}^h$  with such components satisfies (9). At the same time, we realize that the map just constructed belongs neither to the type  $\pi_2$  nor  $\pi_3$ . Under this mapping, straight lines of the space  $A_n$ , given by parametrizations  $x^h = a^h + b^h t$  where  $t$  is a parameter, are mapped onto parabolas in  $\bar{A}_n$  given by the equations

$$\bar{x}^h = D^h + E^h t + F^h t^2$$

where  $D^h = \frac{1}{2} C_\alpha^h (a^\alpha - C^\alpha)^2, E^h = \frac{1}{2} C_\alpha^h (a^\alpha - C^\alpha) b^\alpha,$  and  $F^h = \frac{1}{2} C_\alpha^h (b^\alpha)^2$ . The only exceptions come for those straight lines for which the vectors  $E^h$  and  $F^h$  happen to be collinear: if this is the case the image of such a line is a straight line again.

Finally let us note that the equations (10) generate a system of almost geodesic maps of type  $\pi_1$  of a flat space if we consider the coefficients  $C_i^h, C^h$  and  $x_0^h$  to be continuous parameters.

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